The Abacus Made Easy

Second Edition

A Simplified Manual for Teaching the Cranmer Abacus

by

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ABOUT THE AUTHOR

Dr. Mae E. Davidow had been a teacher at Overbrook School for the Blind from 1935 to 1971. This gifted teacher, however, first came to Overbrook as a student, having lost her sight at the age of 10. She received a B.A. degree from New Jersey College for Women, now Douglas College, part of Rutgers University. Temple University granted her a Master's Degree in 1949 and then a doctorate in 1960.

At Overbrook, Dr. Davidow was instrumental in establishing the use of the Cranmer Abacus as a part of the regular curriculum. Her enthusiasm for this pioneer method of teaching mathematics led others to adopt the use of the abacus. In her role as coordinating teacher, she worked with the members of the Mathematics Department and the results were highly successful. Hopeful that this success at Overbrook might be experienced by many teachers elsewhere, she was encouraged to write this manual.

We are indeed appreciative of the unstinting effort of Dr. Davidow in presenting *The Abacus Made Easy*.

Joseph J. Kerr
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ACKNOWLEDGMENTS

In the writing of this manual I have received invaluable aid and suggestions from the many teachers and students with whom I have worked on the use of the abacus.

I owe a great debt of gratitude to the administrators of the Overbrook School for the Blind who encouraged me to delve more deeply into the study of the abacus, and who made it possible for me to attend the first Abacus Institute ever held in America. I appreciate the opportunity that was given to me to work with the instructors of mathematics at the Overbrook School as they were teaching the use of the abacus to their students. In this way both teacher and student learned the language and method of operation.

I feel it is appropriate to acknowledge with gratitude the work of Fred Gissoni, whose text, **Using the Cranmer Abacus**, helped lay the foundation and groundwork which enabled me to write this manual.

I wish to express my special thanks to the many volunteers who gave unstintingly of their time to work directly with me in compiling this manuscript.

In particular, the generous work of my friend, Leona Wendkos, should be recognized with sincere gratitude. Miss Wendkos was formerly Head of Science Department of John Bartram High School, Philadelphia, subsequently Supervisor in the Intern Teaching Program at Temple University, is currently engaged in part-time

supervision in the Student Teaching Program at St. Joseph's College, Philadelphia.

I wish to thank this tireless worker for her patience, hard work, and keen unerring eye in the revision of this edition. *The Abacus Made Easy* will be made still easier for the student because of her constant insistence on clarity and repetition for the sake of emphasis and understanding. Without her invaluable assistance and collaboration, this second edition would not have been possible.

Recognition and appreciation are also due Mr. Peeter Vilms of Santa Rosa, California, for this skillful presentation of diagrams which should make the operation of the abacus easier.

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FOREWORD

Using the Cranmer Abacus is an excellent text written by Fred L. Gissoni of the Bureau of Rehabilitation Services, Kentucky Department of Education, at Lexington, Kentucky.

In 1964 I attended the Abacus Institute at the University of Kentucky, held under the direction of Mr. Gissoni. This was the first such institute ever conducted in America. At that time Professor Paske, an instructor of the abacus at the School for the Blind in Copenhagen, Denmark, was a participant who lent his invaluable advice and shared his experience with the group.

In the past two years, I have taught mathematics, using the Cranmer Abacus and Fred Gissoni's text. However, as an instructor in the use of the abacus to both teachers and students, I found it desirable to have a simplified manual with which to work. To this end, I have made, with Mr. Gissoni's permission, adaptations of his text. This has been done while working with teachers and with children from second grade through high school. It is my hope that this manual will make the teaching and the learning of the abacus more meaningful to both students and teachers. I shall endeavor to explain in the simplest, most concise manner how to add, subtract, multiply, divide, and to handle decimals, fractions, percent and square root.

The method of instruction for an individual or for a class is essentially the same. The approach to teaching the use of the abacus is likewise the same for a primary, elementary, junior or senior high school student, a college student, a teacher, a parent, or any interested person, either blind or sighted, who wishes to learn to work with the abacus. Before utilizing the abacus to perform arithmetical operations, one must become familiar with the tool itself.

In teaching the use of the abacus, the teacher must be very patient, remembering to employ the language level of the individual student. The procedures must move slowly at first; the language must be simple; every step must be understood thoroughly. As one step is learned, it must be practiced again and again before moving to the next step. Both the slow learner and the gifted learn the manipulation of the abacus in the same manner. After the initial learning has taken place, and the individual knows how to manipulate the tool, he can operate it at his own speed.

Mae E. Davidow, Ed.D.

PREFACE
to
The Abacus Made Easy
Second Edition

The author feels that the past few years in working with the abacus have enriched her background to such an extent that she is able to make additions, corrections, and deletions to the first edition of her book. Her work with students, and work with teachers whom she taught at different colleges, helped her greatly. The summer workshop that she conducted in Raleigh, North Carolina, under the auspices of Eastern Carolina College, in 1968, and the workshop of Northern College at the South Dakota School for the Blind in Aberdeen, South Dakota, in the summer of 1969, also gave her greater insight into different methods of handling the tool. Subsequently, in the summers of 1972 and 1973, in workshops for teachers at the University of North Dakota, Temple University in Philadelphia, and Virginia University, she gained additional valuable experience. The teachers of blind children, teachers of sighted children, and also administrators in both schools for the blind and public schools furnished excellent hints and suggestions.

The author was privileged to work with slow learners, underachievers, and the gifted in public schools where she found the abacus to be of value to the sighted student as well as to the blind one. The classroom teacher recognized that mathematics became more meaningful and more interesting to the pupils who used this tool. Teachers were amazed at the progress made by their underachievers; administrators who sat in on many of these sessions were also interested in the progress made by students in the short time spent with the tool.

This second edition is introduced with the hope that the abacus may serve as the tool to make the teaching and learning of mathematics easier and more interesting. With visits to more schools and colleges, it is the hope of the writer to demonstrate, further, to teachers and administrators that the abacus can be used to achieve this goal.

Resistance to change is understandable. In this case, change involves the learning and the use of a new tool by students and teachers, devising new teacher in-service courses by administrators, purchasing new books and tools by Boards of Education, and introducing a new tool to parents. The author, herself, experienced change in learning the use of a new tool. Because she, as well as her students, found it difficult to master the abacus with former methods of instruction, she wrote and revised *The Abacus Made Easy* with the hope that this second edition might be a valuable guide in the use of the abacus.

INTRODUCTION TO THE TOOL

The first lessons in the use of the abacus consist in handling the tool in order to become thoroughly acquainted with its parts and structure before working with it.

Preliminary examination shows that the abacus is an oblong frame to which are attached thirteen vertical **rods**. On each rod are five **beads** or counters, which travel up and down. A horizontal **separation bar** cuts across all thirteen rods, separating four of the beads from the single fifth bead.

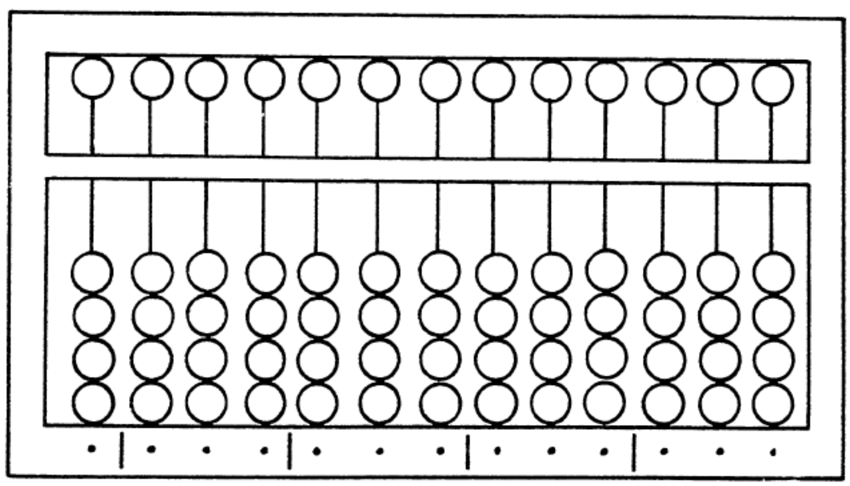


Fig. 1, The Abacus in Zero Position

Further observation should be made with the abacus in its proper operating position as it rests on a horizontal surface, i.e., with the section containing the single beads at the top of the frame, and the section with the four beads on each rod toward the bottom. It is important to maintain the proper hand position, i.e., the index finger of the left hand is always immediately to the left of the index finger of the right hand as the hands move along the frame. Instructions for finding the markings on the frame of the abacus, and the use of the tool in general, follow.

With the tool and the hands in the proper positions, let us first run the fingers along the face of the lower edge of the frame. We find a raised dot (like a pin head) below each rod, and a raised vertical line after every third dot. The dots are used to locate rods needed to write or set numbers; the lines, which are called **unit marks** serve as commas and decimal points. These same dots and lines are found on the separation bar between the groups of four beads below, and the one bead above, the bar.

Now let us place the index finger of the right hand on the dot at the extreme right, on the bottom of the frame. This dot is in line with the rod to the extreme right. This first rod is called the **units** rod or column. Each of the four beads on the first rod at the extreme right, below the separation bar, has a value of 1; the single bead on that rod, above the separation bar, has a value of 5 ones or 5.

Since the abacus is based on the decimal system, each bead below the horizontal bar on every successive rod has a value **10** times that of the corresponding bead on the preceding rod, (moving from right to left). Thus, the second rod from the right is the **tens** column, each bead of which has a value of 10 or 10 ones; the single bead above the horizontal bar on that rod has a value of 5 tens or 50. The next is the **hundreds** column, each bead below the horizontal bar having a value of 100 or 10 tens;

the single bead above the horizontal bar on that rod has a value of 5 hundreds or 500.

Next, in succession from right to left, there is the **thousands** column in which each bead below the separation bar has a value of 1,000 or 10 hundreds; the single bead above the separation bar on that rod has a value of 5 thousands or 5,000. Then follows the **ten-thousands** column in which each bead below the separation bar has a value of 10,000 or 10 one-thousands, and the single bead above the bar has a value of 5 ten-thousands or 50,000. Next, in succession, there are the **hundred-thousands** column, the **millions** column, etc., up to the **trillions** column.

Returning to the examination of the markings on the bottom of the frame, we move along the bottom of the frame from the extreme right to left, with the index fingers in the proper position. (The position of the hands is extremely important in all operations as we manipulate the abacus.) In this way we can see the relationship of each dot to its column, and the use of the vertical lines as commas in writing numbers. The key, while learning the structure and fundamentals of the Cranmer Abacus, is patient repetition.

As an aid to the implementation of the text, drawings for the manipulation of the tool have been prepared by Mr. Peeter Vilms. Diagrams throughout the text show the first setting of beads in the example, followed by steps necessary for its solution. Shaded areas indicate beads which **will move**, or **are to be moved**, in the course of the abacus operation for the calculations illustrated. Generally, the shaded bead indicates the next step in the operation, i.e., the bead (or beads) to be "cleared" or "set". The last drawing in a series usually indicates the final position, or answer, on the abacus.

Whenever further clarification of the drawings used in the solution of a given example is necessary, details will be supplied within the body of the text.

SETTING NUMBERS

When we record a number on the abacus, we do not say we "write" it. In abacus language we use the word "set" instead of "write". When we remove or erase a number, we say that we "clear" it. When beads are moved toward the separation bar to set a number, the beads acquire a value. Each bead below the separation bar has a value of one. The single bead above the separation bar, on any rod, has a value of five. When all the beads on a rod are moved in either direction as far away as possible from the separation bar, they lose their value and the number is **cleared**. The rod is then in **zero position**.

On a single rod (in a single column) we can set the ten digits 0 through 9. To show 0, the rod, or column, should be in zero position, as explained above. To set 1, with the rod, or column, in zero position, we move the top bead below the separation bar **up** to the bar. To set 2, we clear the 1 by moving the bead down, and then move two beads up to the bar; to set 3, we clear the 2 and move three beads upward; to set 4, we clear the 3 and move four more beads up.

To continue, in order to set 5, we clear the 4 and move the single bead which is above the separation bar **down** to the bar. To set 6, when the column is in zero position, we move the single bead **down** to the separation bar, and move the top bead in the group of four in the same column **up** to the separation bar.

(5 + 1 = 6)

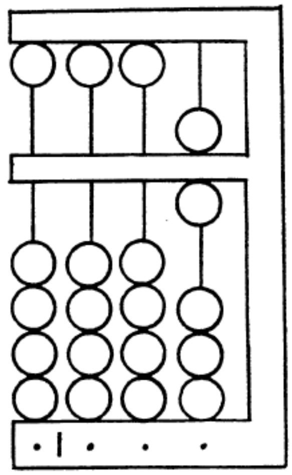


Fig. 2, Setting of 6

To set 7, we set 5 and then set 2 in the same column (5 + 2 = 7). To set 8, we set 5 above the bar and 3 below it. (5 + 3 = 8).

To set 9, we set 5 and then 4 in the same column. In setting 9, we see that all beads on a rod are moved as close to the separation bar as possible.

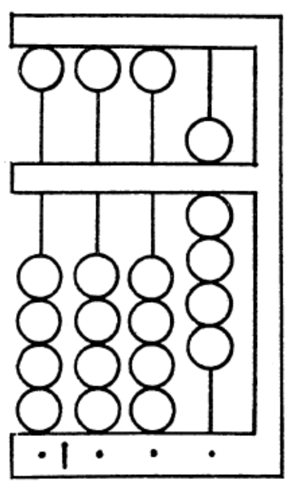


Fig. 3, Setting of 9

Much time must be devoted to setting numbers. As numbers are set with the right index finger, or thumb and index finger, the left index finger always rests on the rod immediately to the left of the right hand. Let us set 12. The right index finger, or thumb, sets a 1 in the tens column (the second column from the right), while the left index finger rests on the rod of the hundreds column. Then both hands move to the right, and the right index finger and thumb push up, or set, two beads in the units column.

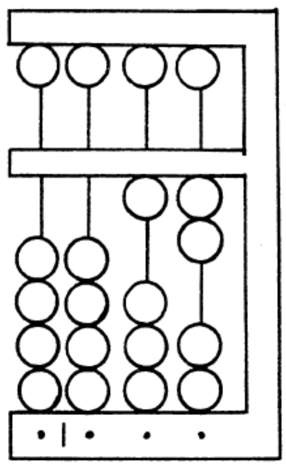


Fig. 4, Setting of 12

Much practice is needed in setting and reading numbers. Small children can set, for example, the number of desks in the classroom, the number of sisters they have, and so on. Older children can set their phone numbers and then have them read by a classmate. Each teacher assigns numbers according to the level of her group. When the student can set and read numbers quickly, he is ready to begin addition.

ADDITION

The operation of addition may be started by adding the number 1 to any number successively. To show this, with the abacus in zero position, we place the index finger of the right hand on the extreme right rod. This is the **units** column. The left hand, as we mentioned before, is immediately to the left of the right hand — in the **tens** column. We set 1 by moving the uppermost bead of the lower beads in the units column **up** to the bar. This is done with the index finger, or thumb, of the right hand. To add 1 to this 1, we move the next bead upward, giving us 2. If we add 1 to this, and then another 1, we have 4 in the units column. This is known as **direct** addition.

If 1 is to be added to 4, it can not be done **directly**, because there are no more lower beads to move up to the bar (Fig. 4a). Therefore, we move the right index finger above the bar and set 5 by pushing the bead **down** to the bar (Fig. 4b). However, we added 5 instead of 1, so that now we must figure out how much more than 1 we added. There are several ways in which this can be done. Each teacher can decide, according to the grade level, what technique to use. One teacher might say, "I asked you to give me 1 cent, and you gave me a nickel. How much change should I give you?" The student figures out that 1 from 5 is 4, and he must "return" 4 cents. We think of the four beads that we have set on the abacus as four pennies, and if we clear the four beads by pushing them down, we are returning the four pennies (Fig. 4c). This is known as indirect addition.

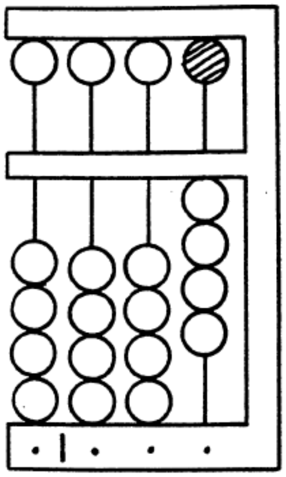


Fig. 4a

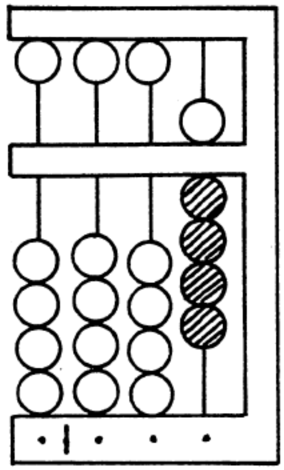


Fig. 4b

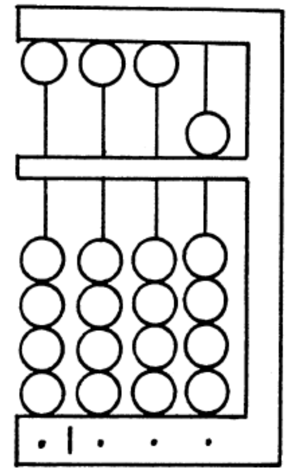


Fig. 4c

Addition of 1 to 4

The different ways in which this can be explained depend upon the age of the student, his understanding of numbers, and also on the resourcefulness of the teacher. One second grade teacher refers to this indirect addition as the working with partners. The child learns that if he wishes to add 1 to 4, and there are no more beads below the bar, he must set a 5 above the bar. He already knows that the partners of 5 are 4 and 1. Since the partners of 4 and 1 are complements of the 5 already set, the child clears the 4 by pushing the four beads down. If, on the other hand, 4 is added to 1, the child sets 5 and clears 1.

After the student learns the relationship of 5 and its partners, 4 and 1, he can set 5 and clear 4 in one continuous motion, resulting in 5 in the units column.

To continue with the addition of 1, we proceed as before. With the 5 set in the units column, we move one bead up to the bar ($5 + 1 = 6$). Setting another bead to add 1 to 6, we have 7. Adding 1 to 7, we have 8; adding another 1 to 8, we have 9. To add 1 more to 9, we find that there are no more beads in the units column to add directly. The left index finger, or thumb, that has been resting in the column immediately to the left of the right hand (in the tens column), now assists because there are no more beads to add with the right index finger (Fig. 5a). We can refer to the left index finger as the "helping finger". The "helping finger" pushes up one bead to the separation bar. This one bead has a value of 10, since it is in the tens column. We needed only 1 more, but the left index finger added 10 more. Therefore, we must determine how many more than 1 we added (Fig. 5b). Again, this can be figured in several ways:

(1) We have 9 and need 1 more, but the left hand added 10. How many more than 1 did we add? 1 from 10 is 9. We have 9 too many, so the right hand clears the 9 (Fig. 5c). We now have 10 in the tens column and 0 in the units column. This gives us a value of 10. The abacus shows that 9 plus 1 equals 10.

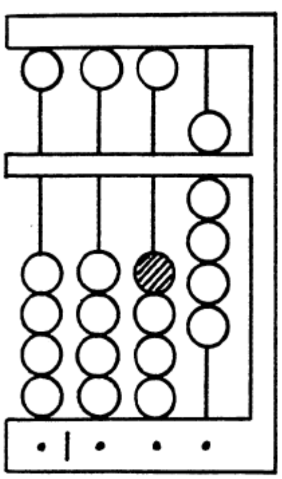


Fig. 5a

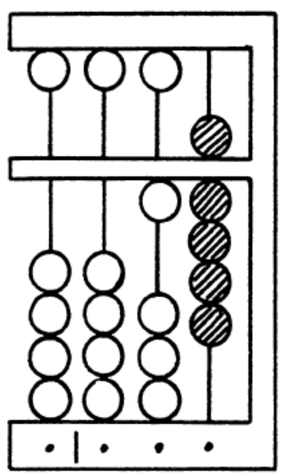


Fig. 5b

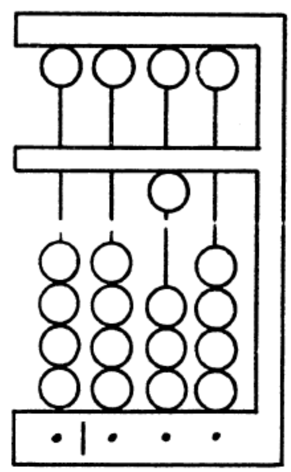


Fig. 5c

Addition of 1 to 9

Note: Steps for the solution of the two foregoing indirect addition example, (4 + 1 and 9 + 1) are shown in a series of three diagrams each. The first drawing in the series shows the original number with the bead of the first operation shaded to indicate the next step; the second drawing in the series shows the interim appearance of the abacus with the shaded bead (or beads) needing further clearing and/or setting; the third shows the sum.

This explanation should suffice for two later examples of indirect addition in this section, viz.: 15 + 6 and 28 + 8.

(2) In terms of money, we needed 1 cent, or a penny, to add to 9 cents. But we added 10 cents, or one dime. Therefore, we have 9 cents too many, so we clear the 9 in the units column.

(3) Again, we can refer to partners. The partners of 10 are 1 and 9. Indirect addition can be performed as above, by setting 10 and clearing 9.

As we continue to add 1's, we add them in the units column with the right index finger or thumb. 1 in the units column added to 10 in the tens column gives us 11. When we add another bead in the units column, we have 12 and so on. For further practice, we can continue to add 1 until we have set 44, remembering that the beads in the units column have values from 1 to 9, and the beads in the tens column have values from 10 to 90. Continuing with the addition of 1, we set 5 and clear 4 in the units column, showing 45 on the abacus.

If we continue to add 1, we reach 49. Then, with the left index finger ("helping finger") we set 5 and clear 4 in the tens column, and with the right index finger we clear 9 in the units column. The abacus now shows 50. Again we can continue adding 1 until we have 99. We find that we can not add 1 to 99 with the right hand in the units column because all the ones have been used; neither can we add a 10 with the left hand in the tens column because all the tens have been used. Therefore, the left hand moves to the hundreds column and pushes up one bead which has a value of 100, or 10 tens. This gives us not only 9 extra ones but also 9 extra tens. Thus, the left index finger clears the 9 tens, and the right index finger clears the 9 ones. This is because, although only 1 was to be added to 99, 10 tens were added, or 100, thus making it necessary to clear the extra 99.

A good addition exercise to use after the introductory practice is to add the numbers 1 through 9 consecutively. First, let us add 1 to 0. We begin with the hands in the starting position — the right hand in the units column, and the left hand immediately to the left of it in the tens column. With the right index finger or thumb, we set 1. Then, to add 2, we move up two more beads in the same column, giving us 3. We are now ready to add 3 more. Since there is only one bead left below the separation bar in the units column, we utilize the bead above the bar. We set 5. How many more than 3 did we set? 3 from 5 is 2. Since we set two too many, we clear 2. The abacus shows 6. If we are now to add 4 more, we find that we have only three beads left in the units column, so we must resort to indirect addition. The left hand will have to assist. The left hand sets 10. How many more than 4 did we add? 4 from 10 is 6, so we clear 6 with the right hand, giving us 10 on the abacus. We are now ready to add 5 to 10, which can be done directly by setting 5 with the right hand, giving us 15 on the abacus.

To continue: — 6 can not be added to 15 directly (Fig. 6a), since there are only four beads left in the units column, so the left hand assists by setting another ten. That is 4 too many, so we must clear 4. We can not clear 4 directly with the right hand, so the right hand clears 5 and sets 1 ($5 - 4 = 1$), (Fig. 6b). To make it more meaningful, we can say, "We wanted to take away 4 pennies, but we took away a nickel, so we owe 1 penny." Therefore, we set 1. We now have 21 on the abacus (Fig. 6c).

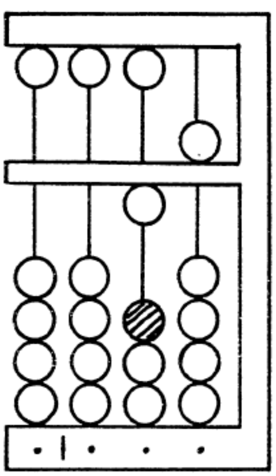


Fig. 6a

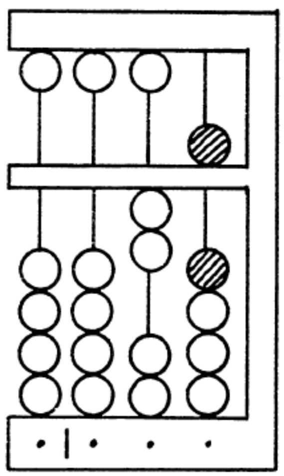


Fig. 6b

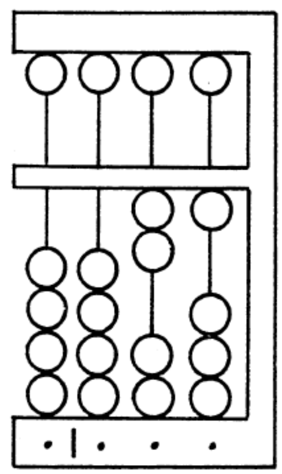


Fig. 6c

Addition of 6 to 15

7 can be added to 21 directly by adding 5 and 2 more in the units column, giving us 28 (Fig. 7a). 8 can not be added to 28 directly, so the left hand assists and sets a

ten (Fig. 7b). 8 from 10 is 2, so the right hand clears 2, resulting in 36 on the abacus (Fig. 7c).

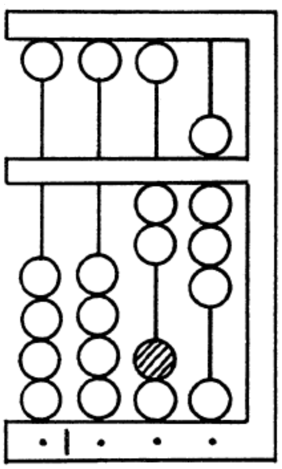


Fig. 7a

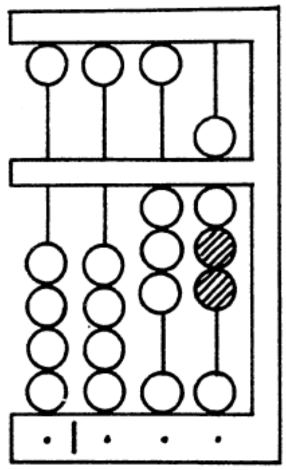


Fig. 7b

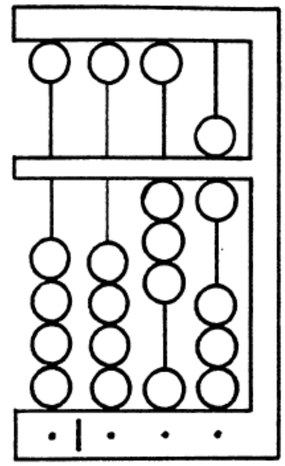


Fig. 7c

Addition of 8 to 28

To this, 9 can not be added directly, so here, again, the left hand assists and sets a ten. 9 from 10 is 1, so the right hand clears 1. The abacus now shows 45.

For additional practice, 1, 2, 3, 4 etc. may be added consecutively beginning with 45.

EXERCISES

(a) 3 + 5 = _____ (d) 42 + 4 = _____

(b) 9 + 3 = _____ (e) 35 + 7 = _____

(c) 5 + 8 = _____ (f) 53 + 6 = _____

ANSWERS

(a) 8 (d) 46

(b) 12 (e) 42

(c) 13 (f) 59

ADDITION OF MULTIPLE-DIGIT NUMBERS

Addition of numbers having two or more digits differs from addition of "pencil and paper arithmetic" in the following ways:

(1) Only **one** addend is at first set on the abacus.

(2) Addition is performed from **left to right** on the abacus.

(3) Each digit is added to the digit in the corresponding column as the addition proceeds from left to right.

(4) It is not necessary to "carry" because the left hand assists.

For example, when 27 is added to 36, the procedure is as follows:

The right hand sets 3 of 36 in the tens column, while the left hand rests immediately to the left of it in the hundreds column. Then both hands move to the right, the right hand sets 6 in the units column, and the left hand rests on the 3 in the tens column. (For low-vision students, and sighted students and teachers, it is helpful to keep the "pointer" fingers on the frame below the appropriate columns instead of resting the fingers on the rods. In this manner the numbers are not covered, making it possible to read them with the eye.)

After setting 36, the right hand moves to the tens column in order to add the 2 of 27 to the 3 of 36. Since 2 can not be added to 3 **directly**, the right hand sets 5 and clears 3 (2 from 5 is 3). Both hands move to the right to add 7 of 27 to 6 of 36. Since 7 can not be added **directly**, the left hand assists by setting a ten. 10 is 3 more than 7, so the right hand clears 5 and sets 2 in the units column (3 from 5 is 2). The answer is 63.

Addition of three-digit numbers which can be added **directly** is not difficult. A simple example is 123 plus 321. The 1 of 123 is set in the hundreds column with the right hand, (the left hand resting immediately to the left of it). Both hands move to the right, and the right hand sets 2 in the tens column. Then both hands move to the right again to set the 3 in the units column. To add 321 to 123, the right hand adds the 3 of 321 to the 1 of 123 in the hundreds column. Then both hands move to the right, and the right hand adds 2 in the tens column. The right hand moves again to the right and adds 1 in the units column. The answer is 444. In this case, the left hand did not have to assist the right hand.

In order to add a number such as 789 to 444, the addition is done **indirectly**. After setting 444, the right hand is placed on the 4 in the hundreds column. Since 7 can not be added to 4 directly, the left hand assists by setting a ten in the thousands column. This ten is equal to 10 hundreds; but only 7 hundreds are to be added, so 3 hundreds are cleared with the right hand (7 from 10 is 3). Both hands then move to the right in order to add 8 tens. Since 8 tens can not be added directly, the left hand assists by setting one bead in the hundreds column. This bead is equal to 10 tens, so 2 tens too many have been added. Therefore, the right hand clears 2 tens. Both hands move to the right again to add 9. Since 9 can not be added directly, the left hand assists again by setting a ten in the tens column. Since this adds 10 instead of 9, the right hand clears 1 in the units column. The answer is 1233.

EXERCISES

(a) 23 + 25 = _____	(d) 425 + 63 = _____
(b) 26 + 165 = _____	(e) 473 + 192 = _____
(c) 16 + 72 = _____	(f) 395 + 827 = _____

ANSWERS

(a) 48	(d) 488
(b) 191	(e) 665
(c) 88	(f) 1222

SUBTRACTION

Subtraction can be introduced at the same time that addition is being taught. While addition is the **sum** of two or more numbers, subtraction is the **difference** between two numbers. To perform subtraction on the abacus, we take away, or **clear**, beads on one rod at a time, working, as in addition, from **left** to **right.**

For example, when we add 1 to 1 on the abacus, we get 2. If we subtract, or take away, 1 from that 2, we **clear** 1 and have 1 left. As another example, if we add 1 to 3 on the abacus, we set 3, add 1, and get 4. To subtract 2 from that 4, we take away, or **clear**, 2, giving us 2.

Note: — In setting and clearing beads, we must always be mindful of the hand position, i.e., the left hand always follows the right.

As suggested above, subtraction may be taught in conjunction with addition. The following may serve as an illustration: — With the abacus in zero position we add 1 to zero by setting 1. To subtract 1 from that 1, we **clear** 1. The answer is zero. If we add 1, and then 2 more to zero, we get 3. Subtracting (clearing) 1, we have 2 left; continuing with the subtraction of 2, the result is zero. Beginning again, we set 1, add 2 and get 3. Attempting to add 3 more to that 3, we find we can not do so directly, so we set 5 which is 2 too many. (We recall that the partner of 5 and 3 is 2, or, "money-wise", that if we do not have 3 cents we can use a nickel which is 2 cents more than 3 cents.) Therefore, we clear 2. The last addition gives a sum of 6.

To continue with the process of subtraction: — If we subtract, or take away, 1 from 6, we clear 1 and have 5 remaining. In subtracting 2 from 5, we find that we can not clear 2 directly, but we can clear 5, which is 3 too many, so we "return", or set 3. The difference between 5 and 2 is 3. 2 can be subtracted from 3 directly by clearing 2 and leaving 1. 1 can be subtracted, or taken away, from 1 directly, giving the answer zero.

To reinforce what has already been learned, let us start again from the zero position on the abacus, add 1, then 2 more which gives us 3. To add 3 to that 3, we must resort to indirect addition by setting 5 and clearing 2, resulting in 6. We see that we can not add 4 to 6 directly with the right hand, so the left hand helps by setting a ten which is 6 too many ($10 - 4 = 6$), so we clear 6 with the right hand, leaving 10 on the abacus. The **sum** of 6 and 4 is 10.

To turn again to subtraction, let us take away 1, 2, 3, 4 consecutively from 10 on the abacus. First, 10 minus 1: — To subtract 1, we put the right hand in the units column, with the left hand immediately to the left of it. Since there is a zero in the units column, the left hand must assist by clearing the ten in the tens column. A check shows that we have cleared 9 too many ($10 - 1 = 9$), so we set 5 and 4 in the units column. 2 can be subtracted from 9 directly by clearing 2, leaving 7. However, 3 can not be subtracted from 7 directly, but the right index finger can clear 5, which is 2 too many. The partner of 5 and 3 is 2, so we clear 5 and set 2, giving us 4. The last step in a case of **indirect** subtraction. Subtracting 4 from 4 gives us zero.

Let us work the following subtraction example involving two-digit numbers: — 32 minus 21.
With the right hand we set 3 of 32 in the tens column. Then both hands move to the right to set the 2 of 32 in the units column. From this, we are to take away, or subtract, 21. We recall that, in subtraction, as in addition, we work from left to right. So, with the right hand on the 3 of 32, and the left hand immediately to the left of it, we subtract, or clear, the 2 of 21 from the 3 of 32, giving us 1 in the tens column. Then both hands move to the right, and the right hand takes away, (clears), or subtracts 1 of the 21 from the 2 of the 32, giving us 1 in the units column. The answer is 11. This is a case of **direct** subtraction.

Let us work an example of **indirect** subtraction involving two-digit numbers: — 42 minus 29.
With the abacus in zero position, we set 4 of 42 in the tens column and 2 of 42 in the units column. Then, with the right hand on the 4 of 42, and the left hand immediately to the left of it, we subtract 2 of 29 from 4 of 42, giving us 2 in the tens column. Both hands move to the right to subtract 9 of the 29 from the 2 of 42 in the units column. But since 9 can not be subtracted from 2 directly, the left hand assists by clearing, or subtracting, a ten in the tens column. Since 10 is 1 more than 9 to be subtracted, we have subtracted one too many, so we "return", or set, 1 in the units column, giving us 3. The answer is 13.

A good reinforcement exercise is to add 1 through 9 consecutively until 45 is the result, (See section on ADDITION) then to subtract 1 through 9 consecutively from 45. With 45 on the abacus, let us begin by subtracting **1** from 45. With the right hand in the units column and the left hand immediately to the left of it, in the tens column, we find we can not subtract 1 from 5 directly, so we clear 5 and set 4, giving us 44 on the abacus.

2 can be subtracted from 44 directly, resulting in 42 on the abacus. Now to subtract **3** from 42. The right hand can not subtract 3 from 2, so the left hand assists by clearing a ten which is seven too many (since the partner of 3 to give us 10 is 7). Therefore the right hand sets 7 (5 and 2) in the units column where there are already two ones. The abacus now shows 39 from which we subtract **4**. This can be done directly by clearing 4 with the right hand, giving 35 on the abacus. **5** can be subtracted directly by clearing 5 of 35, leaving, on the abacus, 30 from which we are to subtract **6**. The right hand can not subtract 6 because there is a zero in the units column, so the left hand assists by clearing a ten, leaving two tens in the tens column. But we cleared 4 too many (the partner of 6 to give us 10 is 4), so we set 4 in the units column, giving us 24 on the abacus.

Now we are ready to subtract **7** from 24. Since we can not subtract 7 from 4 in the units column, the left hand assists by clearing, or subtracting, a ten in the tens column. Since only 7 was to be subtracted, we subtracted 3 too many, so we set 3 in the units column. However, we can not set 3 in the units column directly, so we set 5. Checking, we find that we set 2 too many, so the right hand clears 2, giving 7 in the units column. We now have 17 on the abacus.

In subtracting **8** from 17, we see that 8 can not be subtracted from 7, so the left hand again assists by subtracting, or clearing, a ten in the tens column. Since only 8 was to be subtracted, we subtracted 2 too many, so we must set 2 in the units column where we already have 7, giving us 9. **9** can be subtracted from 9 directly, resulting in zero.

The foregoing drill may be reinforced by other exercises, viz.: — Start by setting 1 on the abacus. To this add 1, 2, 3, 4, 5, 6, 7, 8, 9 consecutively until 46 is reached. Then subtract 1 through 9 consecutively until 1 is left on the abacus. The same procedure can be used beginning with 2 on the abacus.

Let us examine a few subtraction examples: — (a) 64 minus 46.

To set 64, the right hand sets 6 in the tens column while the left hand is on the rod immediately to the left of it. Both hands move to the right to set the 4 in the units column (Fig. 8a). Working from left to right, we attempt to subtract 4 from 6 in the tens column, but find we can not do so directly. So we clear 5 with the index finger of the right hand and set 1 with the thumb. This leaves 2 in the tens column (Fig. 8b). Both hands move to the right to subtract 6 from 4 in the units column. Since we can not take away 6 from 4 in the units column, the left hand assists by clearing a ten with the pointer finger. But 10 is 4 more than 6, so 4 ones must be set. However, this cannot be done directly, so we set 5 which is one too many. Therefore, we clear 1 which results in 8 in the units column. The difference between 64 and 46 is 18 (Fig. 8c).

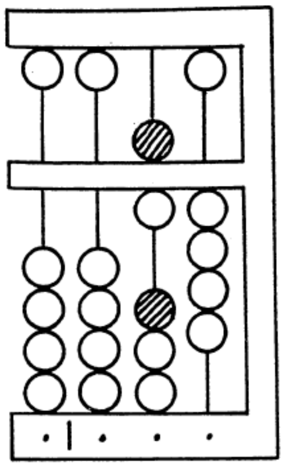


Fig. 8a

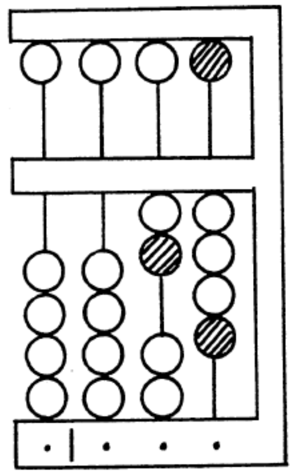


Fig. 8b

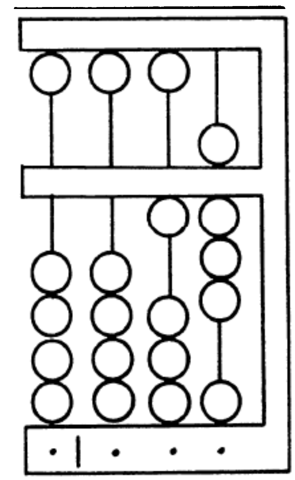


Fig. 8c

Subtraction of 46 from 64

(b) 182 minus 89.

After setting 182 (Fig. 9a), the right hand subtracts 8 of 89 from the 8 of 182, giving us zero in the tens column (Fig. 9b). Both hands move to the right preparatory to subtracting 9 of the 89 from 2 of the 182 in the units column. Because 9 can not be taken away from 2, the left hand should assist by clearing a ten. But there is no ten in the tens column, so the left hand moves to the hundreds column to clear one bead which is equal to 100, or **ten** tens. Since it was necessary to clear one ten, and we cleared ten tens, we must "return", or set, nine of the ten tens in the tens column and keep the extra ten in mind. We are now able to subtract the 9 of 89 from that extra ten (which we had cleared in the hundreds column). Since ten is one more than nine, we "return", or set, 1 in the units column where there are already two ones. The answer is 93 (Fig. 9c).

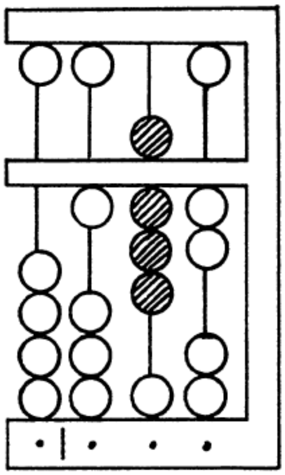
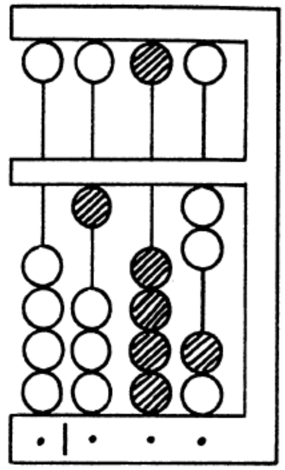
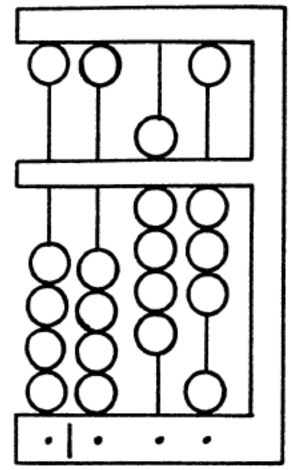
  

Fig. 9a **Fig. 9b** **Fig. 9c**

Subtraction of 89 from 182

EXERCISES

(a) 24 − 12 = _____

(b) 76 − 35 = _____

(c) 82 − 19 = _____

(d) 125 − 93 = _____

(e) 275 − 184 = _____

(f) 378 − 179 = _____

ANSWERS

(a) 12

(b) 41

(c) 63

(d) 32

(e) 91

(f) 199

MULTIPLICATION

Since multiplication is a rapid form of addition, the teacher can introduce multiplication when a number is added to the same number in series more than once. For instance, if 2 is added to 2 on the abacus, it can be seen that two 2's are 4; the addition of 2 more results in 6, and so on. The pupil may have learned at an earlier age to count by 2's, but now he can see how he gets 4, 6, 8, etc. The mathematics becomes more meaningful.

Before the actual performance of multiplication, the following terms must be learned:

Multiplicand — the number which is multiplied

Multiplier — the number by which the multiplying is done

Factors — another name for multiplicand and multiplier

Product — the answer obtained by the multiplication of the factors

On the abacus, the factors can not be set **beneath** each other, as in multiplication on paper. Instead, the multiplier is set on the extreme **left** side of the abacus, and the multiplicand is set in specific columns on the **right** side. The specific position of the multiplicand is determined by the addition of the number of digits in the multiplier, the number of digits in the multiplicand, and "one for the abacus", the sum of which indicates the rod, counting from right to left, on which the first digit of the multiplicand is to be set. A check for the correct placement of the multiplicand is to count the number of unused rods to the right of the multiplicand. This should equal the number of digits in the multiplier only, plus "one for the abacus". The product then appears on the extreme right.

Note: In the examples which follow, the letter "X" stands for the word "times" and indicates the number of times a number is multiplied by another. Thus, 2×4 is interpreted as 2 times 4, where 2 is the multiplier and 4 is the multiplicand.

Besides the placement of the factors, consideration must be given to the zero if, and when, it occurs. There are two important points to observe in this connection, as will be illustrated later.

1. When we multiply one digit by one digit, allowance must always be made on the abacus for **two** digits in the answer. For example, $6 \times 2 = 12$; $1 \times 3 = 03$; $7 \times 3 = 21$; $2 \times 4 = 08$; $0 \times 1 = 00$.

2. To record a zero, we touch, or press, a rod gently (as in addition or subtraction), and then move to the next rod.

A third point to consider in multiplication is the **clearing** of the digits in the multiplicand. After a digit in the multiplicand is multiplied by all the digits in the multiplier, it is **cleared** with the right hand.

To illustrate the placement of the factors, the treatment of the zero, the clearing of the digits in the multiplicand, the following examples may be used:

(a) The first example is 2×4. The multiplier, 2, is set to the extreme left. Adding 1 (for the one digit in the multiplier) and 1 (for the one digit in the multiplicand) and 1 ("for the abacus") results in 3. Therefore, the multiplicand, 4, is set on the third rod from the extreme right. A check for the correct placement of the multiplicand is to count the number of **unused** rods to the right of the multiplicand. This should equal the number of digits in the multiplier, plus "one for the abacus", giving us, in this case, two.

Let us continue with the multiplication of example (a) above: With the right forefinger on the rod on which the multiplicand, 4, is set, and the left forefinger immediately to the left of it, we say 2 $\times$ 4 is zero 8, (08), keeping in mind that allowance must be made on the abacus for two digits in the answer. Both hands move to the right, and the right hand gently touches the rod to record the zero. Both hands move to the right again, and the right hand sets the 8. The right hand clears the multiplicand, 4, since its multiplication by the multiplier, 2, has been completed. The **product,** or answer, is 8.

(b) To further clarify the placement of the multiplicand, we may use the example 3 $\times$ 6. Adding 1 (for the one digit in the multiplier) and 1 (for the one digit in the multiplicand) and 1 ("for the abacus") results in 3. Therefore, the multiplicand, 6, is set on the third rod from the extreme right. Checking the correct placement of the multiplicand, we count the number of **unused** rods to the right of the multiplicand. This should equal the number of digits in the multiplier plus "one for the abacus", giving us, in this case, two. The multiplier, 3, is set at the extreme left. (Fig. 10a)

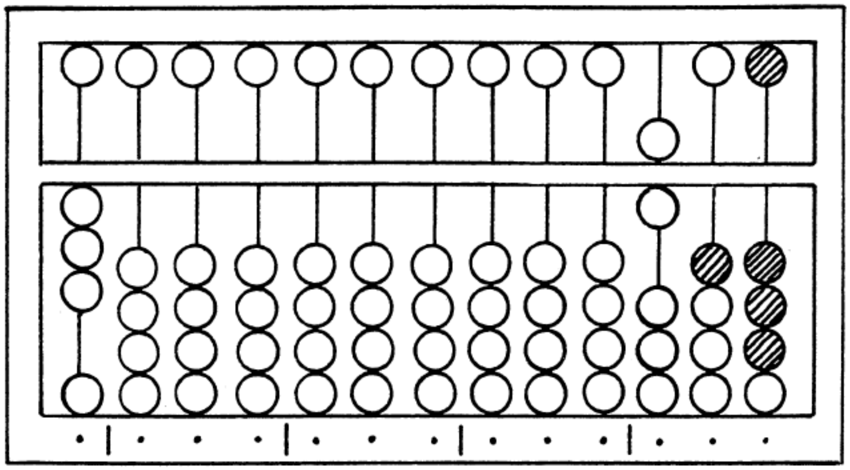
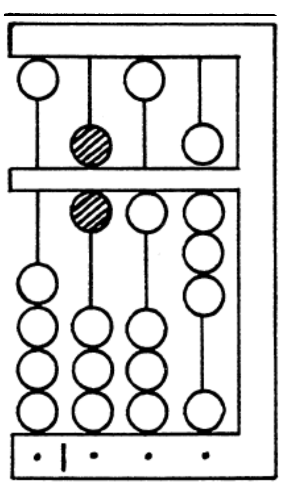


Fig. 10a
Location of Multiplier, 3, and Multiplicand, 6

We proceed with the multiplication as follows: With the right forefinger on the rod on which the digit 6 is located, and the left forefinger immediately to the left of it, we say 3 × 8 is 18. Both hands move to the right, and the right hand sets the 1 of the 18 in the column immediately to the right of the 6. (Fig. 10b.) Both hands move to the right again, and the right hand sets the 8 of the 18 in the next column. Then the right hand clears the multiplicand, 6. The product, 18, is on the extreme right of the abacus. (Fig. 10c)



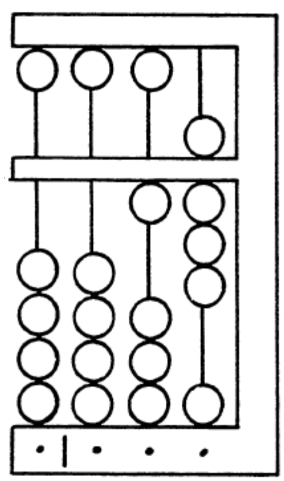


Fig. 10b **Fig. 10c**
Multiplication of 3 × 6

Note: Diagrams for the solution of multiplication examples show, first, the **entire** abacus with the example properly set, and shaded beads to be manipulated in the first operation. Then, each successive step (with the appropriate beads shaded) is shown — including the clearing of each digit of the multiplicand as a separate step.

(c) When there is more than one digit in the multiplicand, the placement of that factor is the same as described before; however, the actual multiplication process requires another step. Let us use the example 9 × 54. To determine the placement of the multiplicand, 54, we add 1 (for the one digit in the multiplier) and 2 (for the two digits in the multiplicand) and 1 ("for the abacus"). This results in 4. Therefore, the 5 of 54 is set on the fourth rod from the extreme right, followed by the 4 of 54 on the third rod. A check for the correct placement of the multiplicand, as explained before, is advisable. The multiplier, 9, is set on the extreme left rod of the abacus (Fig. 11a).

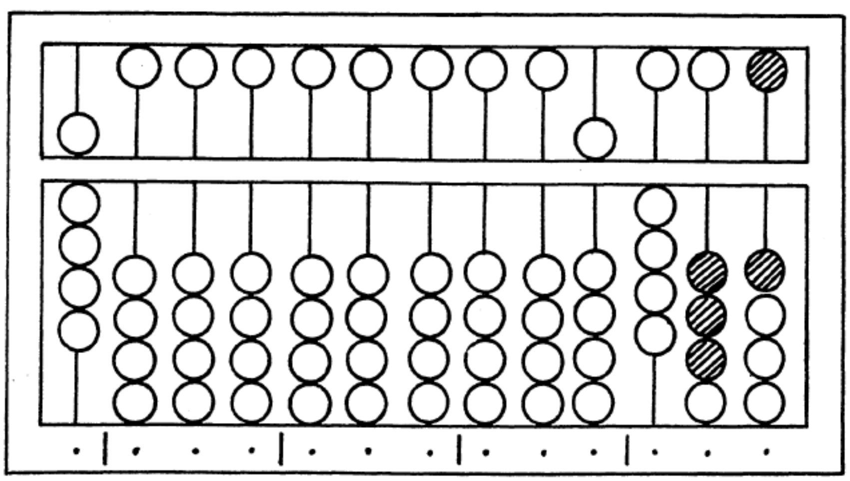


Fig. 11a
Multiplier, 9, on Extreme Left;
Multiplicand, 54, on 4th and 3rd Rod from Right

As usual, in the process of multiplication, the hand position is extremely important. When there is only one digit in the multiplier and two or more digits in the multiplicand, the operation proceeds by working with the digits in the multiplicand **from right to left** and **clearing each digit in the multiplicand after its use.**

Thus, with the right hand on the rod where the 4 is set, and the left hand on the rod where the 5 is set, we say 9 × 4 is 36. Both hands move to the right, and the right hand set the 3 of 36 immediately to the right of the 4 of 54 (Fig. 11b). Both hands move to the right again, and the right hand sets the 6 of 36 in the next column, (the units column). (Fig. 11c) Since the operation using the 4 of 54 has been completed, the right hand clears the 4.

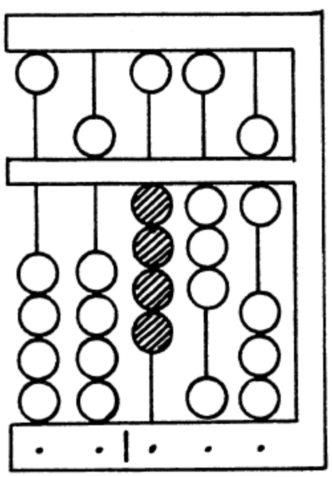


Fig. 11b

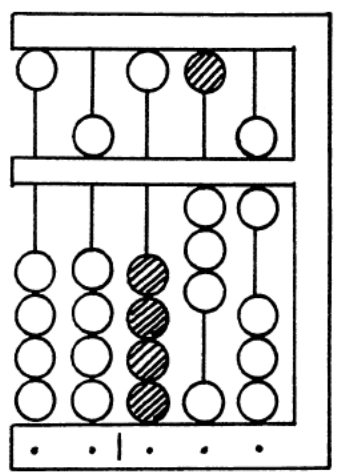


Fig. 11c

Continuing with the multiplication, the right hand is placed on the 5 of 54, the left hand resting on the rod immediately to the left of it, and we say 9 × 5 is 45. Both hands move to the right, and the right hand sets the 4 of 45. Both hands move to the right again, and the right hand sets the 5 of 54 on the same rod as the 3 of 36 (Fig. 11d). The right hand clears the 5 of 54 (Fig. 11e). The product, or answer, is 486.

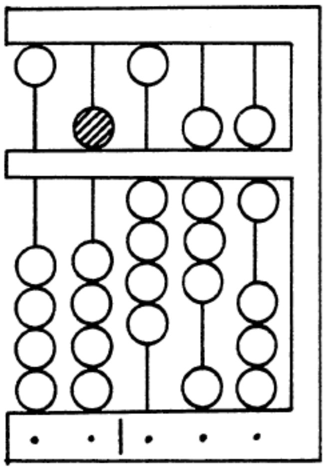


Fig. 11d

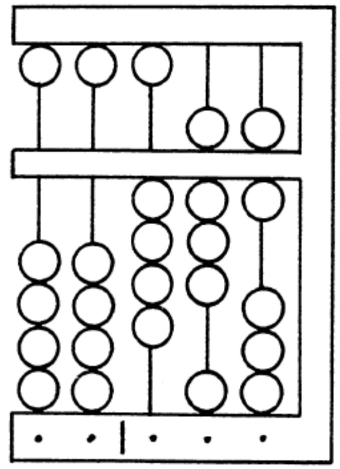


Fig. 11e

Multiplication — 9 × 54

(d) If the factors of the preceding example are reversed, so that there are **two** digits in the multiplier, that is, 54 × 9, we proceed with the process of multiplication by following these rules:

1. The digits of the multiplier are used **separately** from left to right, regardless of the number of digits in the multiplier.
2. The function of the hands, after the use of each digit in the multiplier, is to "hold" the finger on the last used rod to designate the placement of the next digit as the multiplication proceeds.

To return to the multiplication of 54 × 9, the multiplier, 54, is set to the extreme left, and the multiplicand, 9, is set in accordance with the usual practice. Adding 2 (for the two digits in the multiplier) and 1 (for the one digit in the multiplicand) and 1 ("for the abacus") results in 4. The multiplicand, 9, is therefore set on the fourth rod from the right. Following Rule 1 above, the multiplicand is multiplied by each digit of the multiplier separately, in the order of its occurrence, from left to right; these digits are then cleared. The right hand is placed on the multiplicand, 9, and the left hand is placed immediately to the left of it, and we say 5 × 9 is 45. Both hands move to the right, and the 4 of 45 is set immediately to the right of the 9. Both hands move to the right again, and the right hand sets the 5 of 45. The hands are **held** in that position while we proceed with the multiplication of the second digit of the multiplier, 4 × 9 is 36. The right hand sets the 3 of 36 on the same rod as the 5 of 45. Both hands move to the right, and the right hand sets the 6 of 36. The right hand clears the multiplicand, 9, which has been multiplied by the digits of the multiplier. The product is 486.

(e) Another illustration involving treatment of the zero can be found in the examples which follow. To avoid difficulties with zeros, the two important points made earlier must be recalled: — (1) To record a zero, we must touch or press the rod gently, as though we wrote it; (2) when we multiply one digit by another digit, there are always **two** digits in the answer — for example, 6 × 2 is 12, 1 × 3 is 03, 7 × 3 is 21, 4 × 2 is 08, 2 × 0 is 00.

Let us multiply 3 by 304. We add 3 for the digits in our multiplier, 1 for the digit in our multiplicand, and 1 "for the abacus", resulting in 5. We therefore count 5 columns from the extreme right to left, setting our multiplicand, 3, in the fifth column. We set the multiplier 304 to the extreme left. Now we are ready to multiply. Placing the right hand on the 3 of the multiplicand, we say "the 3 of the multiplier, 304, times the 3 in the multiplicand is zero 9 (09)." As we say zero, the right hand moves to the right, and the 0 is recorded by gently touching the rod to the immediate right of the multiplicand. Then both hands move to the right, and the right hands sets the 9 in the next column. We then **hold** the hands in the position as we say "the 0 of 304 times 3 is zero 0 (00)." As we say the first zero, we press the rod where the 9 is; we then move both hands to the right to record the second 0 in the tens place. **Holding** the hands in that position, we say 4 of 304 times 3 is 12. We set the 1 in the tens place, move our hands to the right, and set the 2 in the units place. We then clear the multiplicand, 3, with the right hand. The product is 912.

Let us now multiply 302 by 4, or 4 × 302. After setting the multiplier and multiplicand in their proper places on the abacus, we place the right hand on the 2 and say 4 × 2 is zero 8 (08). Both hands move to the right, and the right hand touches the rod in the tens place to record the 0. Both hands move to the right, and the right hand sets the 8 in the units place. We clear the 2 with the right hand. Since 4 × 0 is 00, there is nothing to clear, so we continue with the multiplication. Placing the right hand on the 3, we say 4 × 3 is 12. Both hands move to the right, and the right hand sets the 1 immediately to the right of the 3. Both hands move to the right again, and the right hand sets the 2. We clear the 3 with the right hand. The product is 1208.

(f) The multiplication of two, three, or more digits in the multiplicand by two, three, or more digits in the multiplier follows the same rules as the preceding examples: It is necessary to remember to multiply each digit of the multiplicand (**starting with the first digit on the right**) by each digit in the multiplier in the order of its occurrence **from left to right**. Then, after each digit in the multiplicand is multiplied by **all** digits in the multiplier, that digit in the multiplicand is cleared with the right hand.

Let us take the example 45 × 67, or 67 multiplied by 45. The multiplier, 45, is set on the extreme left. The multiplicand, 67, is set on the fifth and fourth rods from the right side of the abacus. (Fig. 12a)

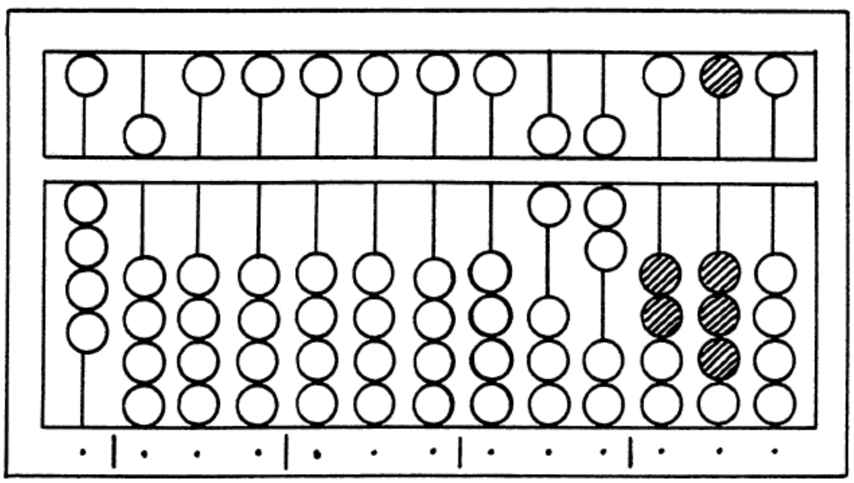


Fig. 12a (45 × 67)
Multiplier on left; Multiplicand on right

With the right hand on the 7 of 67 and the left hand immediately to the left of it on the 6 of 67, we say, the 4 of 45 times the 7 of 67 is 28. Both hands move to the right, and the right hand sets the 2 of 28 immediately to the right of the 7 of 67. Both hands move to the right again, and the right hand sets the 8 of 28 (Fig. 12b). We **hold** the right hand on the 8 while the left hand rests on the 2. The hands are **held** in this position to designate the placement of the first digit in the next step of the multiplication. We are now ready to multiply the 7 of 67 by the 5 of 45, which results in 35. The 3 of 35 is set on the same rod as the 8 of 28. Since we can not set the 3 on the same rod as the 8, the left hand assists by setting a ten and the right hand clears 7, (10 is 7 more than 3). Then both hands move to the right, and the right hand sets the 5 of 35. (Fig. 12c) Since, now the 7 of 67 in the multiplicand has been multiplied by all the digits in the multiplier, the right hand clears the 7 of 67. (Fig. 12d)

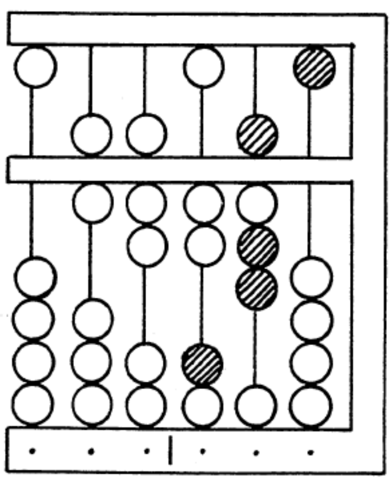


Fig. 12b

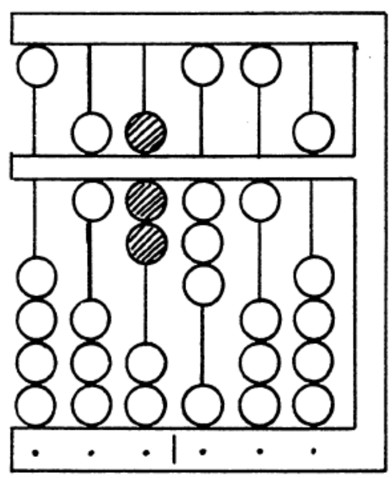


Fig. 12c

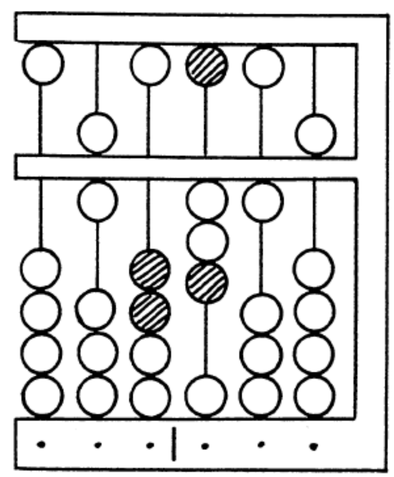


Fig. 12d

The next digit in the multiplicand is to be multiplied by all the digits in the multiplier. Therefore, with the right hand on the 6 of 67, we say 4 times 6 is 24. Both hands move to the right, and the right hand sets the 2 of 24. Both hands move to the right again, and the right hand should set, or add, the 4 of 24 on the same rod as the 3. However, in this case the 4 can not be added to the 3 **directly**, so the right hand sets 5 and clears 1, (5 − 4 = 1), (Fig. 12e). To designate the placement of the first digit in the next step of the multiplication, the right hand is **held** on the rod where the 7 is located, and with the left hand immediately to the left of it, we say 5 times 6 is 30. The 3 of 30 is to be set, or added on the same rod as 7. However, since the 3 can not be set, or added, directly on the same rod, the left hand assists by setting a ten, and the right hand clears 7, (10 − 3 = 7). Both hands move to the right again to set the 0 of 30. To set the 0 on the same rod as the 1, the right hand touches the rod gently (Fig. 12f). Since, now, the 6 of 67 has been multiplied by all the digits in the multiplier, the right hand clears the 6. The product is 3105. (Fig. 12g)

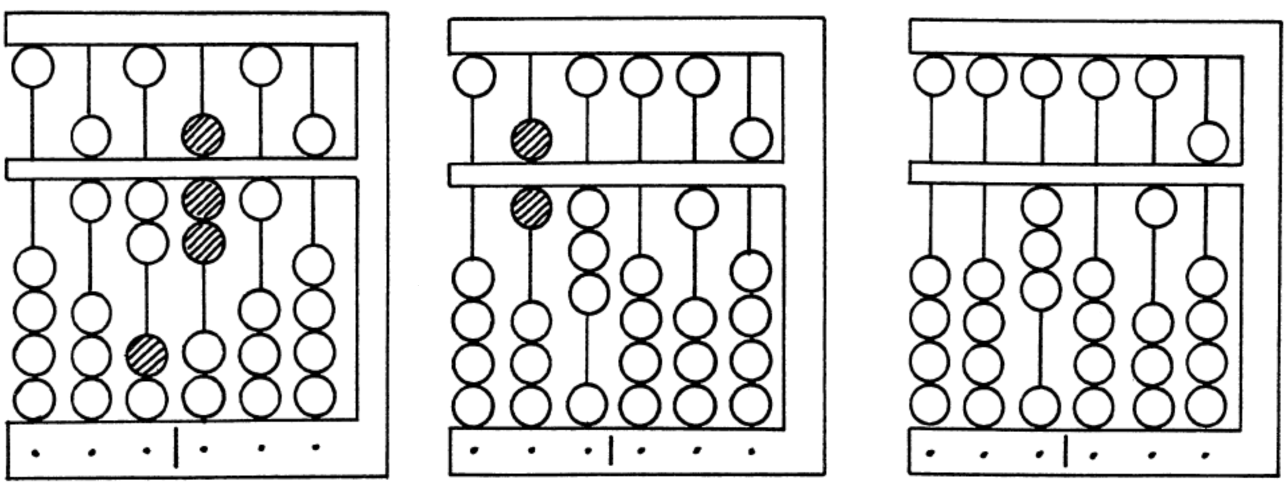


Fig. 12e **Fig. 12f** **Fig. 12g**

Multiplication: — 45 × 67

Multiplication of two or more digits by three or more digits is done in a similar manner. (See the next section.)

EXERCISES

(a) $17 \times 42 =$ _____	(d) $9 \times 49 =$ _____
(b) $56 \times 8 =$ _____	(e) $57 \times 82 =$ _____
(c) $5 \times 203 =$ _____	(f) $307 \times 6 =$ _____

ANSWERS

(a) 714	(d) 441
(b) 448	(e) 4674
(c) 1015	(f) 1842

MULTIPLICATION OF NUMBERS WITH TWO OR MORE DIGITS BY NUMBERS WITH TWO OR MORE DIGITS (With Emphasis on the Treatment of the zero)

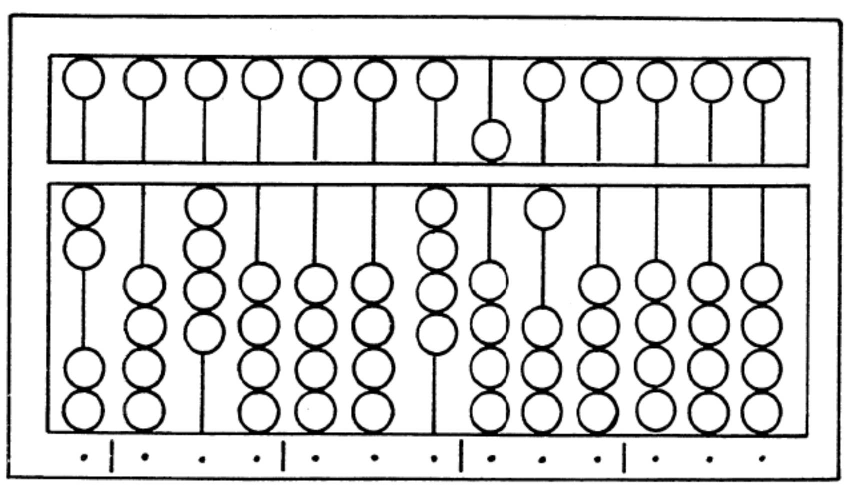
To illustrate multiplication of numbers with two or more digits by numbers with two or more digits, let us use the example 32 × 407. Recalling the locations of the multiplier and multiplicand on the abacus, we set the multiplier, 32, on the extreme left, and the multiplicand, 407, on the sixth, fifth, and fourth rods from the right. With the right hand on the 7 of 407 and the left hand immediately to the left of it, we say 3 of 32 times 7 of 407 is 21. Both hands move to the right, and the right hand sets the 2 of 21. The both hands move to the right again, and the right hand sets the 1 of 21. With the right hand on the 1 and the left hand on the 2, the hands are **held** in that position while we continue with the multiplication of 7 of 407 by the 2 of 32. The product of 2 × 7 is 14. The right hand sets the 1 of 14 on the same rod on which the right hand is resting, where the 1 of 21 in the previous multiplication has been set. Both hands move to the right, and the right hand sets the 4 of 14. Since both digits of the multiplier, 32, have multiplied the 7 of the multiplicand, 407, the right hand clears the 7.

The next step is the multiplication of the 0 of 407 by the multiplier, 32. However, since any digit times zero is 00, it is not necessary to go through the process of multiplying the 0 of 407 by the 3 of 32 and, in turn, by the 2 of 32.

In order to continue with the multiplication, the right hand is placed on the 4 of 407 and the left hand immediately to the left of it, and we say 3 of 32 times 4 of 407 is 12. Both hands move to the right, and the right hand sets the 1 of 12 immediately to the right of the 4; then both hands move to the right again, and the right hand sets the 2 of 12. Both hands are **held** in that position while we continue with the multiplication of the next digit in the multiplier times the 4 of 407. The 2 of 32 times the 4 of 407 is zero 8 (08). The right hand records the 0 of 08 by touching the rod where the right hand is resting. Then both hands move to the right, so that the right hand is ready to set the 8 of 08 on the same rod as the 2. However, 8 can not be added to the 2 **directly**, so the left hand assists by setting a ten. We were to add 8, but we set 10 instead, so the right hand must clear 2, ($10 - 8 = 2$). Since the 4 of 407 has been multiplied by all the digits in the multiplier, the right hand clears the 4. The answer, or product, is 13,024.

Let us now use an example with a zero in the multiplier: 204×451. The multiplier, 204, is set to the extreme left, and the multiplicand, 451, is set on the seventh, sixth, and fifth rods from the right (Fig. 13a).

Fig. 13a (204×451)
Multiplier on left;
Multiplicand on right



Note: Diagrams are provided for this example partly because it is more complex than preceding operations, and partly because it contains a zero in the multiplier. The set-up of the example above is followed later by three diagrams showing the appearance of the working side of the abacus.

With the right hand on the 1 of 451, and the left hand immediately to the left of it, we say 2 of the multiplier, 204, times 1 is zero 2 (02). Both hands move to the right, and the right hand gently touches the rod of the 0 of 02. Then both hands move to the right again, and the right hand sets the 2 of 02. Both hands are **held** in that position while we continue with the multiplication.

When a zero occurs in the multiplier instead of in the multiplicand, as in the preceding example, **every** step in the multiplication must be performed. This is necessary so that the right hand will be **held** on the correct rod for the multiplication of the next digit.

To continue, the 1 of 451 is to be multiplied by the 0 of 204. The product of 0×1 is 00. The first 0 is recorded with the right hand by gently touching the rod on which the right hand is resting. Then both hands move to the right, and the second 0 is recorded by gently touching the next rod with the right hand. The hands are **held** in that position in order to continue with the multiplication of the 1 of 451 by the 4 of 204. The product of 4×1 is zero 4 (04). To record the 0 or 04, the right hand gently touches the rod on which it is resting. Then both hands move to the right, and the right hand sets the 4 of 04, in this case, in the units column. Since the 1 of 451 has been multiplied by each digit of the multiplier, 204, the right hand clears the 1 of 451 (Fig. 13b).

With the right hand on the 5 of 451, and the left hand immediately to the left of it, we say 2 of 204 times 5 of 451 is 10. Both hands move to the right, and the right hand sets the 1 of 10. Then both hands move to the right again, and the right hand records the 0 of 10 by gently touching the rod. Both hands are **held** in that position while we continue with the multiplication. The 0 of 204 times 5 of 451 is zero 0 (00). To record the first 0 of 00, the right hand gently touches the rod on which it is resting. Then both hands move to the right, and the right hand records the second 0 of 00 by gently touching the rod on which the 2 is located. The hands are **held** in that position while we continue with the multiplication. The 4 of 204 times 5 of 451 is 20. The 2 of 20 is set on the same rod as the 2 on which the right hand is resting. Then both hands move to the right, and the right hand records the 0 of 20 by gently touching the rod. Since the 5 of 451 has been multiplied by all the digits of the multiplier, 204, the 5 is cleared (Fig. 13c).

The right hand is then placed on the 4 of 451 and we say 2 × 4 is zero 8 (08). Both hands move to the right, and the right hand gently touches the rod to record the 0 of 08. Both hands move to the right again, and the right hand sets the 8 on the same rod as the 1. The hands are **held** in that position while we continue with the multiplication of the 4 of 451 by the 0 of 204. The product of 0 × 4 is zero 0 (00). The first 0 of 00 is recorded on the same rod as the 9 by gently touching the rod. Then both hands move to the right, and the right hand records the second 0 of 00 by gently touching the rod.

Both hands are **held** in that position while we continue with the multiplication of the 4 of 451 by the 4 of 204. 4 × 4 is 16. The right hand sets the 1 of 16 on the same rod on which the right hand is resting. Both hands move to the right to set the 6 of 16 on the same rod as the 4. But 6 can not be added directly on the same rod as the 4, so the left hand assists by setting a ten. Since the left hand has set 4 too many, the right hand clears 4, (10 − 6 = 4). Since the 4 of 451 has been multiplied by all the digits of the multiplier, the 4 of 451 is cleared. The answer, or product, is 92,004 (Fig. 13d).

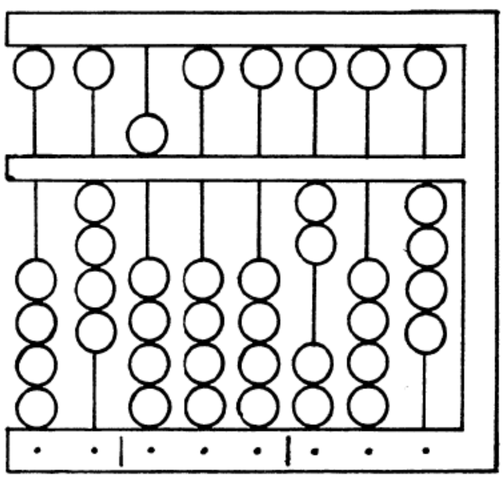


Fig. 13b

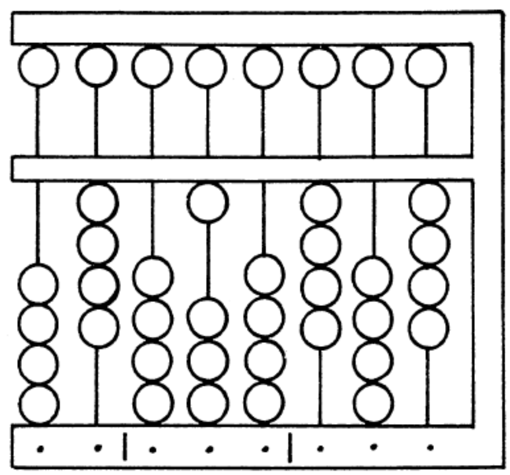


Fig. 13c

Multiplication: — 304 × 451

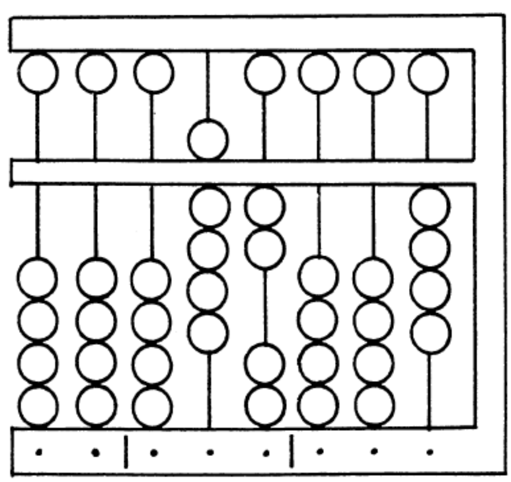


Fig. 13d

At this point, let us use an example in which there is a zero at the end of the multiplicand: 574 × 690. Adding 3 for the digits in the multiplier, and 3 for the digits in the multiplicand, and 1 "for the abacus" results in 7. The multiplicand, 690, is set on the seventh, sixth, and fifth rods from the right, and the multiplier, 574, is set to the extreme left. In such a case, the correct placement of the multiplicand, 690, makes it unnecessary to multiply the 0 at the end of it by all the digits in the multiplier.

With the right hand on the 9 of 690, and the left hand immediately to the left of it, we say 5 of the multiplier, 574, times 9 of the multiplicand, 690, is 45. Both hands move to the right, and the right hand sets the 4 of 45. Then both hands move to the right to set the 5 of 45. With the hands **held** in that position, we continue with the multiplication: — 7 of 574 × 9 of 690 is 63.

The 6 of 63 is set on the same rod as the 5, on which the right hand is resting. But since the right hand can not set a 6 on the same rod as the 5, the left hand assists by setting a ten. The right hand clears 4 by clearing 5 and setting 1 (10 − 6 = 4). Then both hands move to the right to set the 3 of 63. **Holding** both hands in that position, we continue with the multiplication: — 4 of 574 × 9 of 690. The product of 4 × 9 is 36. The right hand sets the 3 of 36 on the same rod as the 3 on which the right hand is resting by setting 5 and clearing 2. Both hands move to the right, and the right hand sets the 6 of 36. Since the 9 of 690 has been multiplied by each digit of the multiplier, 574, the 9 is cleared.

With the right hand on the 6 of 690, and the left hand immediately to the left of it, we say 5 × 6 is 30. Both hands move to the right, and the right hand records the 3 of 30. Both hands move to the right again, and the right hand records the 0 of 30 by gently touching the rod on which the 5 is located. The hands are **held** in that position while we multiply the 6 of 690 by the 7 of 574. The product of 7 × 6 is 42. The 4 of 42 is set on the same rod as the 5. Both hands move to the right, and the 2 of 42 is set on the same rod as the 1. Both hands are **held** in that position while we multiply the 6 of 690 by the 4 of 574. The product of 4 × 6 is 24. the 2 of 24 is set on the same rod as the 3 by setting a 5 and clearing 3. Both hands move to the right to set the 4 of 24 on the same rod as the 6. Since 4 can not be set **directly**, the left hand assists by setting a ten, and the right hand clears 6. Since the 6 of 690 has been multiplied by all the digits in the multiplier, 574, the 6 is cleared. The answer, or product, is 396,060.

EXERCISES

(a) 64 × 295 = ____

(b) 339 × 602 = ____

(c) 48 × 309 = ____

(d) 107 × 834 = ____

(e) 708 × 340 = ____

(f) 123 × 567 = ____

ANSWERS

(a) 18,880

(b) 204,078

(c) 14,832

(d) 89,238

(e) 240,720

(f) 69,741

DIVISION

As we begin to learn division, we must acquaint ourselves with the terms used in a division example. The number that does the dividing is called the **divisor**. The number into which the divisor is divided is called the **dividend**. The answer is the **quotient**. For example, if we divide 48 × 6, the divisor is 6, the dividend is 48, and the quotient is 8.

In setting up an example in division on the abacus, we set the divisor to the extreme **left** and the dividend to the extreme **right**. The quotient will appear to the left of the dividend on specific rods.

In **short** division, there is one digit in the divisor and any number of digits in the dividend. Sometimes the divisor is equal to, or smaller than, the first digit in the dividend; sometimes the divisor is larger than the first digit in the dividend. To determine the placement of the first digit in the quotient on the abacus, the following rules must be observed:

Rule 1: If the divisor is **equal to**, or **smaller than**, the first digit in the dividend, a rod is skipped immediately to the left of the dividend before setting the digit of the quotient.

Rule 2: If the divisor is **larger** than the first digit in the dividend, the digit in the quotient is set immediately to the left of the dividend.

The digit in the quotient is then multiplied by the divisor, and the product resulting from this multiplication is subtracted from the digits in the dividend into which the divisor was divided. This subtraction must always result in a number smaller than the divisor. If not, the division must be continued until the remainder **is** smaller than the divisor. This is known as **upward correction**. The division is then continued using the next digit in the dividend.

Let us now solve the example 792 divided by 4. We set the divisor, 4, to the extreme left, and the dividend, 792, to the extreme right on the third, second, and first rods (Fig. 14a).

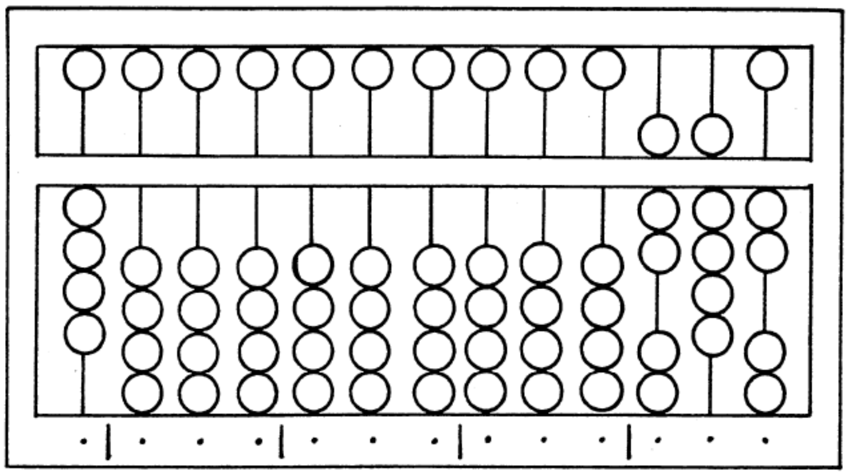


Fig. 14a
Location of Divisor, 4, and Dividend, 792

Note: As in the preceding section, the first set-up of the example is shown, followed later by three successive illustrations of the working side of the tool. These three diagrams show each part of the dividing operation. The quotient appears in the last diagram.

Now we are ready to divide. With the right hand on the 7 of 792, and the left hand immediately to the left of it, we ask "What is the largest number of times 4 will go into 7?" or "How many 4's in 7?" or "What is 7 divided by 4?" The answer is 7 divided by 4 is 1 with something remaining. Where do we place 1, the first digit in the quotient? Since the divisor, 4, is **smaller** than the first digit in the dividend, we follow Rule 1 mentioned previously: If the divisor is equal to, or smaller than, the first digit in the dividend, a rod is skipped immediately to the left of the dividend before setting the digit of the quotient.

Therefore, in this example, both hands move to the left, the right hand gently touches the rod to be skipped (according to the rule), and then both hands move to the left again to set the 1 — in this case, on the fifth rod from the right. Then, with the right hand on the 1, and the left hand immediately to the left of it, we say 1 times the divisor, 4, is zero 4 (04). As we say zero, both hands move to the right, and the right hand records the 0 by gently touching the rod immediately to the right of the 1. Then, as we say 4, both hands move to the right, and the 4 is subtracted from the 7 of the dividend, 792, by clearing 5 and setting 1, giving us 3 (Fig. 14b). We are now ready to divide again. Since the remainder, 3, is smaller than the divisor, 4, we also use the next digit, 9, and we say 39 divided by 4. We think of the largest number of times 4 is contained in 39. (If we can not think of the **largest** number of times, we make use of "upward correction", which will be illustrated later.) The answer is 9 and something remaining.

Where do we place the 9? We compare the divisor, 4, with the first digit of 39. Since the divisor, 4, is **larger** than 3, we follow Rule 2, as mentioned previously: If the divisor is larger than the first digit in the dividend, the digit in the quotient is set immediately to the left of the dividend. Therefore, in this case, the quotient digit, 9, is set immediately to the left of the 3 of 39.

With the right hand on the 9, we say 9 × 4 is 36. Both hands move to the right to subtract the 3 of 36 by clearing the 3 of 39. Then both hands move to the right to subtract the 6 in the column where the 9 is located, leaving 3 (Fig. 14c). We have 32 as the newly acquired dividend, and we say 32 divided by 4 is 8. Since 4, the divisor, is larger than the first digit, 3, of 32, we set the 8 immediately to the left of the 3 of 32. Now, with the right hand on the 8, and the left hand immediately to the left of it, we say 8 times 4 is 32. Both hands move to the right to subtract the 3 of 32 by clearing 3 with the right hand. Then both hands move to the right to subtract the 2. This is done by clearing the 2. Since there is nothing remaining, we see that the original dividend, 792, is evenly divisible by 4 (Fig. 14d). We are now ready to read the answer, which is done in the following manner: — At the abacus, we count one rod for the one digit in the divisor and add "one for the abacus". Everything to the left of that is the quotient. Thus, the quotient is 198 (Fig. 14d). When the dividend is not equally divided by the divisor, i.e., when the division is not exact, there is a remainder which is located on the extreme right of the abacus.

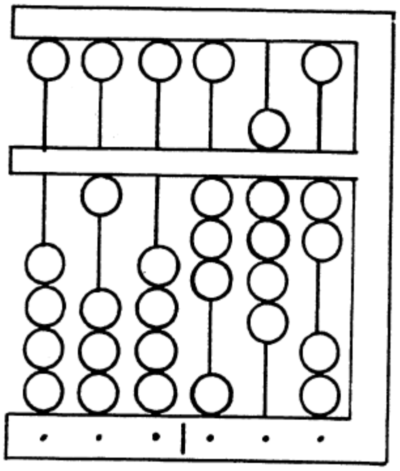


Fig. 14b

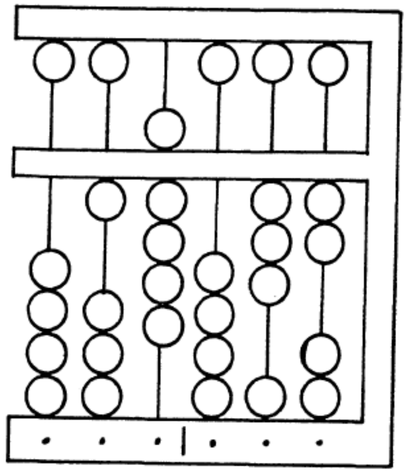


Fig. 14c

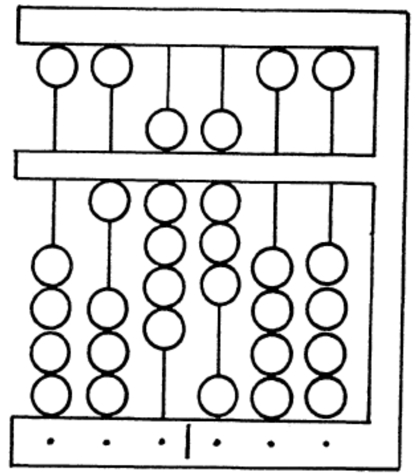


Fig. 14d

Note: As in the preceding section, the set-up of the first example is shown, followed, later, by three successive diagrams of the working side of the tool showing each part of the dividing operation. The quotient appears in the last drawing.

We can now check the division by multiplying quotient by the divisor. The quotient, 198, becomes the multiplicand, and the divisor, 4, becomes the multiplier. We see that these factors are already in the correct location on the abacus for multiplication. With the right hand on the 8 of 198 and the left hand on the 9, we say 4 times 8 is 32. Both hands move to the right, and we set the 3 immediately to the right of the 8. Both hands move to the right again to set the 2. We clear the 8 with the right hand. Placing the right hand on the 9, we say 4 times 9 is 36. Both hands move to the right to set the 3 of 36 to the right of the 9. Then both hands move to the right again to set the 6 of 36. We clear the 9 with the right hand. Then we place the right hand on the 1 and say 4 times 1 is zero 4 (04). As we say zero, both hands move to the right, and the right hand records the 0 by gently touching the rod to the right of the 1. Both hands move to the right again, and the right hand adds the 4 in the same column as the 3, giving us 7. With the right hand we clear the 1. The answer is 792, which is the original dividend. Thus, we see that division is the inverse of multiplication.

Let us work an example in which the division is not exact, to see how to handle a remainder and to use "upward correction", if necessary. We will divide 836 by 9. The divisor is 9 and the dividend is 836, both of which we set in their proper places in preparation for division. We note that the divisor, 9, is larger than the first digit in the dividend, so we use the first two digits and think of the largest number of 9's contained in 83. If, by chance, we think that 7 is the largest number of 9's, we follow Rule 2 and set the quotient, 7, immediately to the left of the 8 of 83. After multiplying 7 by 9, and subtracting 63 from 83, we find that 20 is larger than the divisor, 9. Thus, we see that we did not use the **largest** number of 9's contained in 83 and must make use of "upward correction" by adding to 7 the greatest number of 9's contained in 20. Since 2 is the greatest number of times 9 is contained in 20, 2 must be added to 7 giving us 9 in the quotient. (Upward correction is necessary whenever, in division, a remainder is larger than the divisor.) Multiplying 9 by 2, and subtracting 18 from 20, we now have 26 on the abacus. To continue, the greatest number of 9's contained in 26 is 2 and something remaining. Since the divisor, 9, is larger than the 2 of 26, we set the quotient, 2, immediately to the left of the 2 of 26 (Rule 2). Multiplying 9 by 2, and subtracting 18 from 26, we have 8 on the abacus. Since 8 is smaller than 9, and there are no more digits in the dividend, we are now ready to read the answer. Beginning at the extreme right on the abacus, we count the number of rods equal to the number of digits in the divisor, which in this case is one, and "one for the abacus". Everything to the left of this is the quotient, and anything to the right is the remainder. The answer is 92 and a remainder of 8, or 92 8/9.

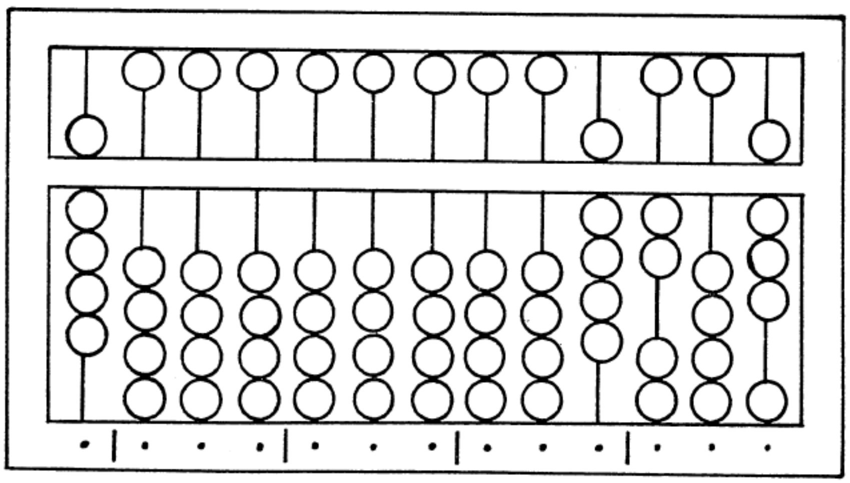


Fig. 15: — 836 ÷ 9
Quotient and Remainder on Right Side

To check this example, we multiply the quotient by the divisor. Placing the right hand on the 2 of the quotient, 92, and the left hand immediately to the left of it, we say the divisor, 9, times 2 is 18. We move both hands to the right to set the 1 of 18 to the right of the 2 of the quotient, 92. Then we move both hands to the right again to set the 8 of 18. But since the remainder, 8, is in the units column, we add 8 of 18 to the remainder by setting a ten with the left hand and clearing 2 with the right hand. The right hand then clears the 2 of 92. Placing the right hand on the 9 in the quotient, we say the divisor, 9, times 9 of the quotient is 81. Both hands move to the right to set the 8 of 81. Then both hands move to the right again to set the 1 of 81 in the column where the 2 is. The right hand clears the 9 of the quotient. The answer is 836, which is the original dividend. Thus, we notice that in checking, the remainder is automatically taken care of.

It is good policy to do the same example over and over, until the process is well established. It is also good practice to divide and then check by multiplication. This gives opportunity for review of division, subtraction, multiplication and addition.

EXERCISES

(a) 255 ÷ 5 = _____	(d) 2112 ÷ 3 = _____
(b) 2419 ÷ 6 = _____	(e) 2348 ÷ 9 = _____
(c) 9870 ÷ 8 = _____	(f) 3340 ÷ 7 = _____

ANSWERS

(a) 51	(d) 704
(b) 403 1/6	(e) 260 8/9
(c) 1233 3/4	(f) 477 1/7

LONG DIVISION

Long division is the process by which a dividend, containing any number of digits, is divided by a divisor containing two or more digits. As the process of long division proceeds, we try to divide the divisor into the same number of digits in the dividend. The example is set on the abacus in the same manner as in short division.

In the example, 495 divided by 45, the divisor, 45, is set to the extreme left, and the dividend, 495, to the extreme right. Since there are two digits in the divisor, we examine the first two digits in the dividend to see how many 45's there are in 49. We see immediately that 45 is contained in 49 one time. Since 45 is smaller than 49, we skip a rod to the left of the 4 in 49 to set the first digit of the quotient. Multiplying 45 by 1, and subtracting it from 49, gives us 4. We now have 45 on the abacus as the partial dividend. We are ready to divide again. 45 is contained in 45 one time. Here again, we skip a rod and set the 1 to the right of the first 1 in the quotient. Multiplying 45 by 1 and subtracting it from 45 results in zero. In order to read the answer, we must determine how many rods to count from the extreme right. Adding two rods for the two digits in the divisor and one rod "for the abacus", we count three rods from right to left. Anything to the left of the 3rd rod is the quotient. The quotient is 11.

The next example will illustrate the use of a **trial** divisor: — 3015 divided by 45

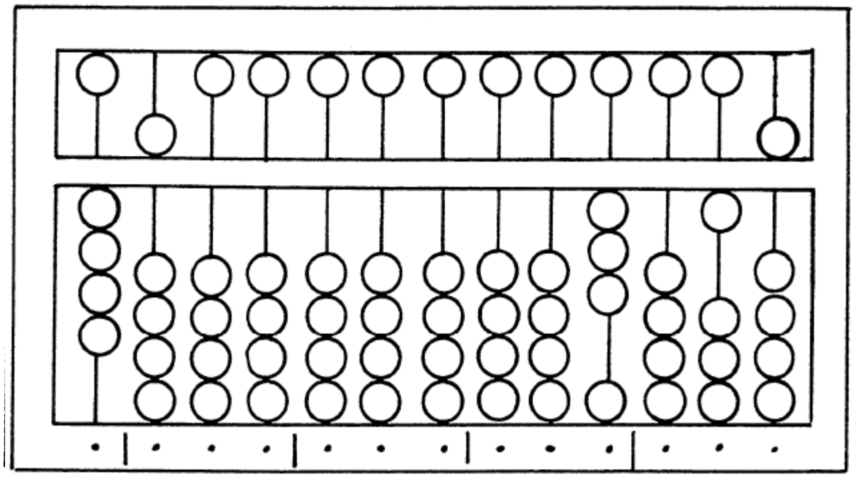


Fig. 16

In the example, 3015 divided by 45, the divisor, 45, is set to the extreme left, and the dividend, 3015, to the extreme right.

When, as in the above example, there is more than one digit in the divisor, and the first digit of the divisor is smaller than, or equal to, the first digit in the dividend, AND, the second digit in the divisor is larger than the second digit in the dividend, it is advisable to use a **trial** divisor. The trial divisor is determined by adding 1 to the first digit of the divisor, 45, making the trial divisor 5.

To return to the example, 3015 divided by 45, we see that the trial divisor, 5, is not contained in the first digit of the dividend. Since the trial divisor, 5, is larger than the first digit in the dividend, we determine how many times it is contained in the first **two** digits of the dividend, i.e., how many times is 5 contained in 30 of 3015? 5 is contained in 30, 6 times. Where to place the 6, the first digit of the quotient? Following Rule 2: — if the first digit of the divisor is larger than the first digit of the dividend, the first digit of the quotient is set immediately to the left of the dividend. Since the 4 of 45 is larger than the 3 of 3015, the 6 is set immediately to the left of the 3.

To continue: — We are now ready to multiply the divisor, 45, by 6, the first digit in the quotient. With the right hand on the 6, and the left hand immediately to the left of it, we say: 6 times 4 is 24. Both hands move to the right, and the right hand subtracts the 2 of 24 from the 3 of 30. Then both hands move to the right to subtract the 4 of 24 from the 0 of 30. Because 4 can not be subtracted from 0, the left hand assists by clearing a ten and the right hand sets 6, since 4 from 10 is 6. With the right hand **held** on that 6, we say: 6 of the quotient times the second digit, 5, of the divisor is 30. The right hand subtracts the 3 of 30 by clearing 5 and setting 2. Then both hands move to the right to subtract the 0 by gently touching the rod. We now have 315 in the partial dividend (Fig. 16a).

We are now ready to divide again. By using the trial divisor, 5, we see that 31 divided by 5 is 6 with something remaining. Since the 4 in the divisor is greater than the 3 of the 31 in the dividend, the 6 is set immediately to the left of the 3 of 31. With the right hand on the 6, (second digit of the quotient), and the left hand immediately to the left of it, we say: 6 times 4, the first digit in the divisor, is 24. Both hands move to the right, and the right hand subtracts the 2 of 24 by clearing 2. Then both hands move to the right again in order to subtract the 4 of 24 from the 1 of 31. Since the right hand can not subtract 4 from 1, the left hand assists by clearing a ten, and the right hand sets 6. This results in 7, on which the right hand is **held**, while the left hand is held immediately to the left of it. We say: 6 times 5, the second digit in the divisor, is 30. The right hand subtracts the 3 of 30 from the 7 by clearing 5 and setting 2. Both hands move to the right, and the right hand records the subtraction of the 0 of 30 by gently touching the rod where the 5 is (Fig. 16b).

We are now ready to divide again. We observe that there is 45 left in the dividend, and note that since the first digit in the divisor is equal to the first digit in the dividend, and the second digit in the divisor is equal to — **not larger** than — the second digit in the dividend, a trial divisor is not necessary. The divisor 45 is contained in the dividend, 45, one time. Where to set the 1? Since the 4 in the divisor 45 is equal to the 4 in the dividend, and the second digit in both are also equal, we must skip a rod to the left of the dividend, and add the 1 in the same column as the 6. With the right hand on the 1 just added, and the left hand to the left of it, we say: 4 times 1 is zero 4, (04). In order to subtract the 0, we move both hands to the right and gently touch the rod with the right hand. Then we move both hands to the right to subtract the 4 of 04 by clearing 4 on the abacus. With both hands **held** in that position, we say: 5 times 1 is zero 5, (05). In order to subtract the 0, we gently touch the rod where the right hand is resting. Then both hands move to the right to subtract the 5 by clearing it.

In order to read the answer, we must determine how many rods to count from the extreme right. Adding 2 rods for the two digits in the divisor and 1 rod "for the abacus", we count 3 rods from right to left. Anything to the left of the 3rd rod is quotient. The quotient is 67 (Fig. 16c).

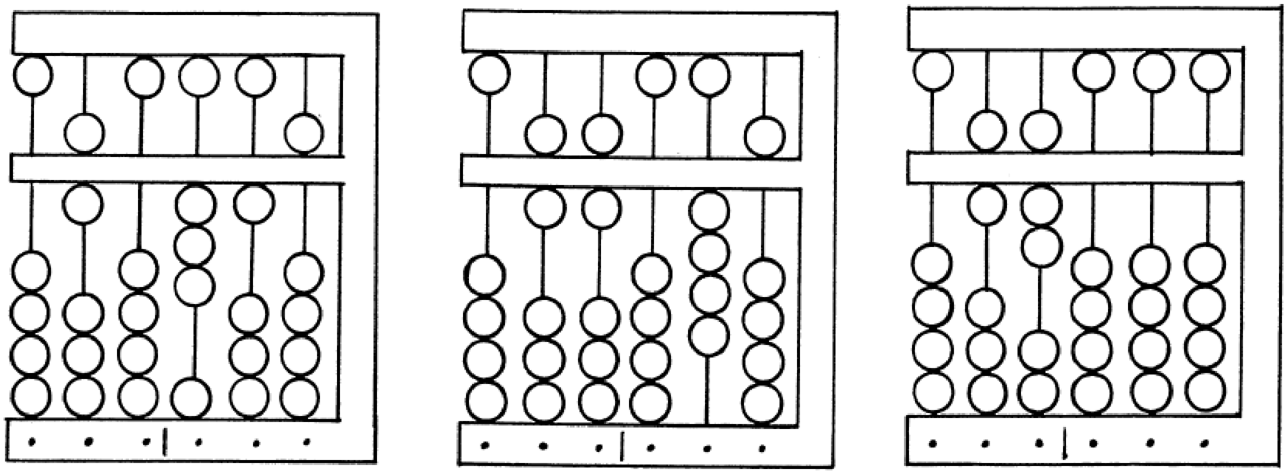


Fig. 16a **Fig. 16b** **Fig. 16c**
Long Division: 3015 ÷ 45

We can now check our answer by multiplying the quotient, 67, by the divisor, 45. The quotient, 67, is located in the proper position for multiplication, i.e., two rods for the two digits in the multiplier, two rods for the two digits in the multiplicand, and 1 rod "for the abacus", verifying that there are 5 places from the extreme right for the location of the 6 of 67. Since the divisor is always set at the extreme left, the 45 is already in that position to be used as the multiplier. It is good policy to check every division example by multiplication, and every multiplication example by division, thus giving the student continued practice in both processes.

Let us work an example with a **remainder**: 1496 ÷ 24. The divisor, 24 is set at the extreme left, and the dividend, 1496, at the extreme right. Since there is more than one digit in the divisor, 24 and 24 is larger than 14, we use a trial divisor obtained by adding 1 to the first digit, 2, of the divisor, giving us 3. 3 is contained in 14, (the first two digits in the dividend), 4 times with something remaining. Since the first digit, 2, in the divisor is larger than the first digit, 1, in the dividend, we set 4 of the quotient immediately to the left of the 1 in the dividend. With the right hand on the 4, and the left hand immediately to the left of it, we say: 4 times 2 is zero 8, (08). Both hands move to the right, and the right hand touches the rod gently to subtract the 0 from the 1; then both hands move to the right again, to subtract the 8 from the 4. Since 8 can not be subtracted from 4 directly, the left hand assists by clearing a ten, and the right hand sets 2 by setting 5 and clearing 3, resulting in 6. With the hands **held** in that position, (the right hand on the 6, and the left hand on the 0), we are now ready to multiply the 4 of the estimated quotient by 4, the second digit of the divisor. 4×4 is 16. The right hand subtracts the 1 of the 16 from the 6 on the rod where the right hand is resting. Then both hands move to the right, and the right hand subtracts the 6 of the 16 from the 9 in the dividend.

We are now ready to divide again. 5 divided by the trial divisor, 3, is 1 and something remaining. Since the 2 in the divisor is smaller than the 5 in the dividend, we skip a rod and set the 1 in the same column where the 4 of the estimated quotient is. This is done by setting 5 and clearing 4. Using that 1, we say: 1 times 2, the first digit in the divisor, is zero 2, (02). We move both hands to the right, and gently touch the rod with the right hand to subtract the 0; then we move both hands to the right again to subtract the 2 from the 5 by clearing 5 and setting 3. With the hands **held** in that position, we say: 1 times 4 is zero 4, (04). We subtract the 0 with the right hand by gently touching the rod on which the 3 is located; then we move both hands to the right to subtract the 4. Since 4 can not be subtracted from 3 directly, we clear a ten with the left hand, and set a 6 with the right, giving us 9, and showing 296 on the abacus as the remaining dividend, and 5 as the partial quotient.

The next step in the division of this example does not require the use of a trial divisor because the 2 in the divisor, 24, is equal to the first digit in the dividend, 29, and the second digit in the divisor is smaller than the second digit in the dividend, so a trial divisor is not necessary because we can see that 24 is contained in 29 one time with something remaining. Where to place the 1? Since the divisor 24 is smaller than 29, (Rule 1), we skip a rod and add 1 to the 5 which is already in the quotient. Multiplying the 1, just added, by the divisor, 24, gives us 24 which we subtract from 29, resulting in 5. We now have 56 on the abacus which is to be divided by 24.

We are now ready to divide again, this time using the trial divisor. 5 divided by the trial divisor, 3, is 1 and something remaining. Since the 2 in the divisor, 24, is smaller than 5 in the dividend, we skip a rod and set the 1. Multiplying 1 by the divisor, 24, and subtracting it from 56, we get 32. We are now ready to divide again: — 3 of 32 divided by the trial divisor, 3, is 1. Since the 2 of the divisor, 24, is smaller than the 3 of 32, we skip a rod, and add 1 on the rod where the other 1 is located, thus making use of "upward correction". With the index finger of the right hand on the 1 which was just added, we say: 1 × 2 is zero 2, (02). Both hands move to the right, and the right hand subtracts the 0 by gently touching the rod. Then both hands move to the right again to subtract the 2 from the 3. With both hands **held** in that position, we say: 1 × 4 is zero 4, (04). The right hand subtracts the 0 by gently touching the rod. Then both hands move to the right to subtract the 4 from the 2. Since 4 can not be subtracted from 2 directly, the left hand assists by clearing a ten and the right hand sets 6, leaving 8 on the extreme right.

To read the answer, we count, from right to left, two rods for the divisor and "one for the abacus". Everything to the left of these **three** rods is the whole number of the quotient, and everything to the right is the fractional remainder. The answer is 62 and 8 remaining, or 62 8/24 or 62 1/3.

We notice, again in long division, that if we have not estimated correctly, the greatest number of times the divisor goes into the dividend, we can always divide another time by the trial divisor as we continue. If a correction is necessary, it is added to the estimated quotient. ("Upward Correction").

Pupils and teachers should do the same example over and over. This is as true in division and multiplication as it is in addition and subtraction.

To recapitulate, the following is a good rule to remember in the placement of the quotient: — (a) When the divisor is compared with the dividend, and the **first** digit in the divisor is **equal** to the first digit in the dividend, and the **second** digit in the divisor is **equal** to, or **smaller** than, the second digit in the dividend, then the divisor will go into the dividend one time and **we skip a rod** before setting the 1. (In this case it is not necessary to use a trial divisor.) (b) If, however, the first digit in the divisor is **equal** to the first digit in the dividend, but the **second** digit in the divisor is **greater** than the second digit in the dividend, we use the trial divisor, but do **not** skip a rod in setting the quotient.

To demonstrate both parts of this rule, let us work this example: — 2639 ÷ 24.
Comparing the first two digits in the divisor with the first two digits in the dividend, and applying part (a) of the rule above, we note that 24 will go into 26 one time. Therefore, we skip a rod and set the 1 in the quotient. Multiplying 24 by 1, and subtracting 24 from 26, results in 2. Thus, 239 appears on the abacus.

Comparing the two digits of the divisor with the 23 of 239, and applying part (b) of the rule above, we do **not** skip a rod in setting the next digit of the quotient. But, in this case, a trial divisor is needed. We recall that the trial divisor is obtained by adding 1 to the first digit in the divisor, making the trial divisor, in this case, 3. Continuing with the operation, 3 is contained in 23 of 239 seven times with something remaining, so we set the 7 immediately to the left of the 2 of 239. Multiplying 24 by 7, and subtracting, results in 71 on the abacus. The trial divisor, 3, is contained in 7 of 71 two times with something remaining. Since the first digit, 2, of the original divisor, 24, is smaller than 7 of 71, we skip a rod and set the 2 on the same rod as the 7. This, again, illustrates "upward correction". We multiply 24 by 2, and subtract, giving us 23.

We are now ready to read the answer. From right to left, we count 2 rods for the divisor, and 1 "for the abacus". Everything to the left of these three rods is the whole number of the quotient, and everything to the right is the fractional remainder. The answer is 109 and 23 remaining, or 109 23/24.

In Fred Gissoni's text, *Using the Cranmer Abacus*, he refers to "upward correction". He also discussed "downward correction". In accordance with this manual, "downward correction" is not used; instead, the use of a trial divisor eliminates the need for "downward correction".

EXAMPLES

(a) 333 ÷ 12 = _____	(d) 4589 ÷ 38 = _____
(b) 9826 ÷ 64 = _____	(e) 3502 ÷ 34 = _____
(c) 8962 ÷ 87 = _____	(f) 63445 ÷ 731 = ____

ANSWERS

(a) 27 9/12 or 27 3/4	(d) 120 29/38
(b) 153 34/64 or 153 17/32	(e) 103
(c) 103 1/87	(f) 86 and 7 remaining

ADDITION OF DECIMALS

The fundamentals of adding and subtracting decimals are the same as those in addition and subtraction of whole numbers. However, first, the placement of the decimal point is to be determined before setting the whole numbers. The comma or "unit mark" is used as a decimal point in both addition and subtraction of decimals. The function of the "unit mark" is described in detail in the first section of the manual, "Introduction to the Tool", P.1.

When we have one, two, or three decimal places in any of the addenda in the example, we use the first decimal point from the right on the horizontal bar of the abacus to set the decimals to the right of it, and the whole numbers to the left of it. Should there be four, five, or six decimal places in the example, we must use the second decimal point from the right. If there are one, two, or three **more** than six decimal places, we use the third decimal point from the right.

For example, let us first add 3.1, 2.04, and 2.005, (3 and 1 tenth, plus 2 and 4 hundredths, plus 2 and 5 thousandths). Since the smallest decimal in this case is thousandths, we need three decimal places. Therefore, we set the first whole number to the left of the first decimal point. Moving from right to left, we find our first decimal place, (the first "unit mark" from the right), and set the whole number, 3, immediately to the left of it, (on the fourth rod from the right). Then to the right of the decimal point we set the 1 to designate .1. We are now ready to add 2.04. We add the whole number 2, to the whole number, 3, which is to the left of the first decimal point. This gives us 5. Next we add .04 to .1 by setting 4 in the hundredths place which is two places to the right of the decimal point. Now we are ready to add 2.005. We add the whole number, 2, to the whole number, 5. Then we add .005 by setting 5 in the thousandths place which is three places to the right of the decimal point. Our answer is 7.145.

Let us now work an example with more than three decimal places, i.e., more than three digits in the decimal.

Example: — 4.0005 + 2.006 + 2.000002 + 3.05 + 10.3 + 1.00002. Since there are more than three digits in some of the decimals in this example, we must use the second decimal point, or "unit mark" from the right as our decimal point.

First, we set the whole number, 4, to the left of the decimal point, (on the seventh rod from the right). Since .0005 (5 ten-thousandths) requires four places after the decimal point, we touch the first three rods after the whole number 4, for the zeros and then set the 5 on the third rod from the right. We then add 2.006 (2 and 6 thousandths) by adding the whole number, 2, to the whole number, 4, resulting in the whole number, 6. To record the two zeros before adding 6 of the .006, we touch two rods immediately to the right of the decimal point, and then add the 6 on the next rod, (fourth rod from the right). We add 2.000002, (2 and 2 millionths), by adding the whole number, 2, to the whole number, 6, resulting in 8; then we touch five rods to the right of the decimal point for the five zeros and we add 2 of the .000002. We now add 3.05 by adding the whole number 3, to the whole number, 8, resulting in the whole number 11; then we move both hands to the immediate right and add .05 by touching the rod to the right of the decimal point for the 0. Both hands move to the right to add the 5 in the hundredths place, which is on the fifth rod from the right. Now we add 10.3 by adding the whole number 10, to the whole number, 11, resulting in the whole number, 21, and adding .3 in the tenths place, (on the sixth rod from the right). Then we add 1.00002, (1 and 2 hundred-thousandths), by adding the whole number, 1, to the whole number, 21, resulting in 22, and adding the 2 of the .00002 by touching four rods for the zeros and adding 2 of the .00002, on the fifth rod to the right of the decimal point. Our answer is 22.356522.

EXAMPLES

(a) 4.09 + 6.209 = _____

(b) 67 + 84.000945 + .592 = _____

(c) 7.00345 + 15.3274 = _____

(d) 6.446 + 6.002 = _____

(e) 21.7765 + 19.32 + 9.08 = _____

(f) 3.2 + 9.934 + .00182 = _____

ANSWERS

(a) 10.299
(b) 151.592945
(c) 22.33085
(d) 12.448
(e) 50.1765
(f) 13.13582

SUBTRACTION OF DECIMALS

We have already mentioned that the subtraction of decimals is done in the same manner as the subtraction of whole numbers. However, we must determine where to place the decimal point. We follow the same procedure as we did in the addition of decimals.

Let us examine a subtraction example: — 5.354 – 2.132 (5 and 354 thousandths minus 2 and 132 thousandths). Both numbers have **three** decimal places. Moving from right to left, we set the whole number 5 to the left of the first decimal point, which is between the third and fourth rods. Then we set .354 to the right of the whole number. We are now ready to subtract 2.132. From the whole number, 5, we subtract the whole number, 2. This leaves 3. From .3 we subtract .1 and this leaves .2. From 5 hundredths we subtract 3 hundredths, and that gives us 2 hundredths. From 4 thousandths we subtract 2 thousandths, which leaves 2 thousandths. Our answer is 3.222. This is an example of **direct** subtraction.

Let us now work an example that has more than three places in the decimal, and which requires **indirect** subtraction in which the left hand assists: — 12.672138 – 5.39467 (12 and 672,138 millionths minus 5 and 39,467 hundred-thousandths). Since the largest number of decimal places is six, the whole number, 12, is set to the left of the second decimal point from the right. The next six digits in the decimal after the 12 are set to the right of the whole number, 12. We are now ready to subtract the whole number, 5, from the whole number, 12. This must be done **indirectly**, with the assistance of the left hand. In order to subtract 5 from 12, we place the right hand on the 2 of the 12 and the left hand on the 1. Since we can not subtract 5 from 2, the left hand assists by clearing a ten and the right hand sets 5 (5 from 10 gives us 5), resulting in the whole number 7. Both hands move to the right in order to subtract the decimal. With the right hand on the 6, we subtract 3 from 6 by clearing 5 and setting 2. Both hands move to the right to subtract 9 from 7. The left hand assists by clearing a ten and the right hand sets 1. Both hands move to the right to subtract 4 from 2. The left hand assists by clearing a ten and the right hand sets 6. Both hands move to the right to subtract 6 from 1. The left hand assists by clearing a ten and the right hand should set 4. Since we can not set 4 directly, the right hand sets 5 and clears 1. Both hands move to the right to subtract 7 from 3. The left hand assists by clearing a ten and the right hand should set 3. We can not set 3 with the right hand, so the right hand sets 5 and clears 2. Both hands move to the right; since there are no millionths to subtract from 8 millionths, the 8 remains on the abacus. The answer is 7.277468.

Thus, we see that **indirect** addition and subtraction examples are worked in the same manner with decimals as with whole numbers.

EXAMPLES

(a) 9.173 − 4.152 = ____

(b) 67.081 − 42.931 = ____

(c) $784.03964 - 522.7432 =$ _____

(d) $78.403 - 59.132 =$ _____

(e) $62.7193 - 54.037 =$ _____

(f) $548.85357 - 28.78538 =$ _____

ANSWERS

(a) 5.021

(b) 24.150

(c) 261.29644

(d) 19.271

(e) 8.6823

(f) 520.06819

MULTIPLICATION OF DECIMALS

In multiplication of decimals, the factors (the multiplier and multiplicand) are set in the same positions as whole numbers in the process of multiplication. However, to avoid unnecessary steps in the multiplication, the zero, or zeros, immediately to the right of the decimal point, and preceding the first other integer is/are disregarded in setting the number provided the zero, or zeros, are not preceded by a whole number: —viz. .003406 is set as if it were 3406; but 15.02 is set as 1502.

After setting the multiplier and multiplicand, and completing the multiplication in the usual manner, the placement of the decimal point must be determined. This is done by adding the number of decimal places in the multiplier to the number of decimal places in the multiplicand, and using the sum to count from right to left for the placement of the decimal point in the product.

For example, let us multiply .37 by .5. We set the 5 of .5, the multiplier, to the extreme left. Then we count four rods from right to left to set the 37 of .37, the multiplicand (two rods for the multiplicand, and one for the multiplier, and "one for the abacus"). With the right hand on the 7, and the left hand on the 3, we say: 5 times 7 is 35. Both hands move to the right and the right hand sets the 3 of 35; then both hands move to the right to set the 5 of 35 and the right hand clears the 7 of 37. With the right hand on the 3 of 37, we say: 5 times 3 is 15. Both hands move to the right, and the right hand sets the 1 of 15. Then both hands move to the right again and the right hand sets the 5 of 15 in the same column as the 3. We clear the 3 with the right hand. The product is 185. Since there is one decimal place in the multiplier, and two decimal places in the multiplicand, the sum of the decimal places is three. So, counting from right to left we point off three decimal places in the product. The answer is .185.

If, in the above example, the multiplicand would have been .037, it would have been set also as 37; if the multiplicand had been 3.07, it would have been set as 307; if the multiplicand had been .00307, it would also have been set as 307. If the multiplier had been .005, it would have been set as 5, and so on. In any case, the position of the decimal point in the product is determined by counting, from right to left, the sum of the decimal places in the multiplicand and the multiplier.

To emphasize, let us multiply 23.4 by .02. We set the multiplier and the multiplicand on the abacus as in the multiplication of whole numbers. To set 23.4, the multiplicand, we count one rod for the 2 of .02 (the multiplier), three rods for the three digits in the multiplicand and one rod "for the abacus". That gives us five rods from right to left. Beginning with the fifth rod, we set 2, 3, 4. To set the multiplier, .02, the 2 is set to the extreme left and we are ready to multiply.

The multiplication is carried on in the usual manner, resulting in 468 on the abacus. Where to place the decimal point? Since there are two decimal places in the multiplier, .02, and one decimal place in the multiplicand, 23.4, the decimal places add up to three. We count three rods from right to left for the location of the decimal point, and so the answer is .468.

EXAMPLES

(a) $5 \times 9.7 =$ ____

(b) $9.85 \times 7.8 =$ ____

(c) $9.308 \times .024 =$ ____

(d) $74.3 \times 14.03 =$ ____

(e) $54.05 \times 8.07 =$ ____

(f) $7.005 \times 9.02 =$ ____

ANSWERS

(a) 48.5

(b) 76.830

(c) .223392

(d) 1042.429

(e) 436.1835

(f) 63.18510

DIVISION OF DECIMALS

In division of decimals, we treat the operation as division of whole numbers. To set the divisor and dividend, consideration of zeros is the same as in setting factors in multiplication of decimals. Then, to determine the number of places to point off in the quotient, we **subtract** the number of decimal places in the divisor from the number of decimal places in the dividend. This gives us the number of places to point off from right to left. However, if there are more decimal places in the divisor than there are in the dividend, we add as many zeros to the dividend as are needed to make the number of decimal places equal to the number of decimal places in the divisor. Then, by subtraction as indicated above, we find there will be no decimal places to point off in the quotient. If there are no decimal places in the dividend, it is understood that a decimal point follows a whole number. We then add as many zeros as are needed to make the number of decimal places in the dividend equal to the number of decimal places in the divisor. Again, there will be no decimal place in the quotient. However, if we add more zeros in the dividend to carry out the answer to a decimal, the number of decimal places in the quotient is determined as stated above.

Let us take the example 8.4 divided by .4. We set the 4 to the extreme left, and the 8 and 4 to the extreme right. 8 divided by 4 is 2. Since the divisor, 4, is **smaller** than the 8 in the dividend, we skip a rod and set the 2. Then with the right hand on the 2, we say: 2×4 is zero eight (08). As we say zero, both hands move to the right, the right hand gently touches the rod, then both hands move to the right again, and the right hand subtracts the 8 by clearing it.

We are now ready to divide again. 4 divided by 4 is zero 1 (01). Since 4, the divisor, is **equal** to the 4 in the dividend, we skip a rod, placing the 1 to the right of the 2 in the answer or quotient. With the right hand on the 1 and the left hand immediately to the left of it, we say: 1 × 4 is zero 4 (04). As we say zero, both hands move to the right, and the right hand touches the rod gently. Then both hands move to the right again and the right hand subtracts the 4 by clearing it. We are now ready to read the answer. Since there is one digit in the divisor and we add "one for the abacus", we count two rods from right to left. Everything to the left of this is the answer. We have 21 on the abacus. Where to place the decimal point? Because there is one decimal point in the divisor, and one decimal point in the dividend, we subtract 1 from 1 and we get zero. Therefore, we do not point off any places in the answer. The answer, or quotient, is 21.

Let us take the example 96, divided by .3. Since there is one decimal place in the divisor, and there are no decimal places in the dividend, we annex as many zeros in the dividend as there are decimal places in the divisor, recalling that every whole number is immediately followed by a decimal point. It is necessary to annex those zeros to give as many decimal places, or more, in the dividend as there are in the divisor in order to make subtraction possible, and to obtain the correct number of decimal places in the quotient.

Now to work the example, 96 ÷ .3. According to the explanation given in the paragraph above, we annex one zero to the dividend, making it 960 and set it to the extreme right on the abacus. We set the 3 of the divisor to the extreme left. 9 ÷ 3 is zero 3 (03). Since the 3 in the divisor is **smaller** than the 9 in the dividend, we skip a rod and set the 3. With the right hand on the 3, we say: 3 × 3 is zero 9 (09). Both hands move to the right, and the right hand gently touches the rod to record the 0. Then both hands move to the right again and the right hand subtracts the 9. We are now ready to divide again. 6 ÷ 3 is zero 2 (02). Here again, the 3 in the divisor is smaller than the 6 in the dividend; so we skip a rod and place the 2 to the right of the 3 in the quotient. With the right hand on the 2, we continue with the multiplication and subtraction in the usual manner. We are now ready to read the answer. Since we count one rod for the one digit in the divisor, and one rod "for the abacus", we count two rods from right to left, and everything to the left of these rods is the answer. The answer is 320. Now to find where to place the decimal point: — There was one place in the divisor and we added one place in the dividend. By subtracting the number of decimal places in the divisor from the number in the dividend, we find that 1 − 1 = 0. Therefore, there are no decimal places in the quotient. Our answer remains 320.

Fred Gissoni, in his text *Using the Cranmer Abacus*, describes a completely different method for locating the decimal point. Interested persons may consult this reference.

EXERCISES

(a) $.205 \div .5 =$ _____

(b) $25.25 \div 2.5 =$ _____

(c) $43.56 \div .71 =$ _____

(d) $96.3 \div 3 =$ _____

(e) $370.37 \div .11 =$ _____

(f) $27.24 \div 3.26 =$ _____

ANSWERS

(a) .41

(b) 10.1

(c) 61.35+

(d) 32.1

(e) 3367

(f) 8.35+

FRACTIONS

A fraction is a numeral which is part of a number, such as 1/2 or 3/4. In the introduction of fractions, it is necessary to distinguish between the **numerator** and the **denominator,** which are called **terms** of the fraction. These terms are separated by a fraction bar. For example, in the common fraction 1/2, 1 is the numerator and 2 is the denominator. The numerator, 1, indicates the number of fractional units taken. The denominator, 2, tells into how many equal parts the unit is divided. When the number in the numerator and the number in the denominator are the same, (viz.: — 3/3, 4/4, 15/15), the value is 1. This shows that the number 1 may be written in several ways depending upon the number of equal parts the whole was divided.

To set a fraction on the abacus, we set the numerator to the extreme left, skip a rod, which **designates the fraction bar**, after which we set the denominator. In order to write 2/2 on the abacus, we set the numerator, 2, to the extreme left, skip a rod, and set the denominator 2, immediately after. Since it is advisable to reduce a fraction to its lowest terms, we must first find the greatest common factor of both the numerator and the denominator, and then divide it into each term. (The greatest common factor is the greatest number which is contained evenly in both the numerator and denominator). In this case, the greatest factor common to 2, the numerator, and to 2, the denominator, is 2. Therefore by division, 2/2 = 1/1 = 1.

Let us find the simplest name, or simplest form for

4/8. We set the numerator 4 to the extreme left, skip a rod and set the denominator 8. What is the largest common factor of 4 and 8? It is 4. The numerator, divided by 4 is 1, so we change the 4 to 1. The denominator, 8, divided by 4 is 2, so we change the 8 to 2. Thus, the simplest form for 4/8 is 1/2. This process is commonly referred to as **reduction of fractions**.

EXAMPLES

(a) 3/6 = ____

(b) 8/14 = ____

(c) 8/36 = ____

(d) 7/21 = ____

(e) 10/16 = ____

(f) 5/25 = ____

ANSWERS

(a) 1/2

(b) 4/7

(c) 2/9

(d) 1/3

(e) 5/8

(f) 1/5

ADDITION OF FRACTIONS

In setting up the abacus for the addition and the subtraction of fractions, we divide the tool into three sections. One section is for the whole number, if there is one, the second section is for the numerator, and the third section is for the denominator.

Since a **common** denominator is necessary in the addition and subtraction of fractions, we must first determine what it will be, and then place it in the denominator section. The common denominator is placed to the extreme right; the whole number is placed to the left of the **second** unit mark (from the right on the abacus). If there is one digit in the whole number, it is placed on the seventh column from the right; if there are two digits, the eighth and seventh columns, and that order are used, and so forth. The numerator is placed to the left of the **first** unit mark from the right. The denominator, as mentioned before, is placed to the extreme right. As an example, a diagram showing 24-3/16 is given below.

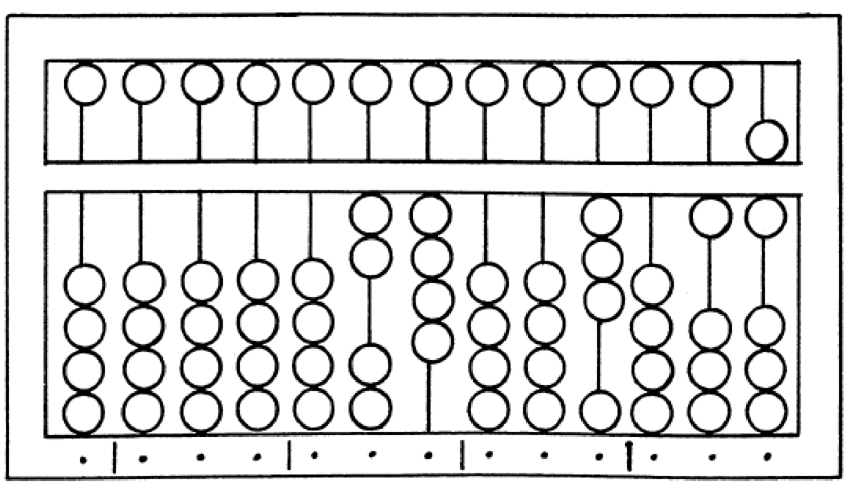


Fig. 17
24-3/16 on Three Sections of the Abacus

If the denominator should be three digits or more, a system must be worked out whereby the three sections are moved to the left and arranged in such a manner that they do not run into the numerator. In like manner, if the numerator is composed of three or more digits, they should be arranged so that they do not run into the whole number.

Since we need a common denominator in the addition and the subtraction of fractions, we often obtain it by multiplying the two denominators together. Another way is to try to find the smallest number into which each denominator can be divided evenly. We must multiply both numerator and denominator by a number that will give a common denominator. Multiplying a numerator and denominator of a fraction by the same number gives an equivalent fraction, but does not change the value. We then place the common denominator in the denominator section of the abacus.

We are now ready to work an example: 4-5/6 + 5-7/9. We must first find the **lowest** common denominator. One way to do this is by the trial and error method: — multiplying the largest denominator by successive numbers beginning with 2 until we find the smallest number into which all the denominators can be divided evenly. In this case, 9 × 2 is 18. We check to see if the other denominator, 6, can be divided evenly into 18. It can; so we use 18 as the lowest common denominator. The denominator, 18, is placed to the extreme right, which is the denominator section.

To continue with the example, we set the whole number, 4, of the 4-5/6 in the whole number section, which is to the left of the second unit mark (the seventh column). In order to change 5/6 to 18ths, we set the 5 to the extreme left, skip a rod, and set the 6. We divide the denominator, 6, into the common denominator, 18, mentally, and we get 3. Then we multiply both terms of the fraction by 3, giving us 15/18. (Multiplying both the numerator and the denominator by the same number does not change the value of the fraction.) We set the numerator, 15, in the numerator section of the abacus, to the left of the first unit mark. The denominator is already in the denominator section at the extreme right.

Let us add the 5-7/9 to the 4 and 15/18. We add the whole number 5 to the 4 in the whole number section giving us the whole number 9. Then we change the 7/9 to 18ths. The denominator, 9, is contained in the common denominator, 18, two times. Multiplying the numerator, 7, by 2, we get 14, which we add to the numerator, 15, already in the numerator section. We now have 29 in the numerator. In examining the numerator and denominator in the fraction 29/18, we find that the numerator is larger than the denominator. This is an **improper** fraction, so we divide the numerator by the denominator, giving us 1 and something remaining. The whole number, 1, is added to the whole number, 9, resulting in 10 in the whole number section. Since we added 1, or 18/18 of the 29/18 to the whole number section, we have 29/18 minus 18/18 or 11/18 left. This leaves us 11 in the numerator section. As we look at the three sections, we find that the answer is 10-11/18.

To review and emphasize the steps taken to change a fraction to an equivalent fraction, the same example will be used: 5/6 changed to 18ths.

Since the denominator, 6, is being changed to the denominator, 18, 18 is set in the denominator section to the extreme right. The numerator, 5, in the fraction 5/6 is set to the extreme left. We skip a rod and set the denominator, 6. With the right hand on the 6, we ask: 6 is contained in the common denominator, 18, how many times? The answer is 3. We clear the denominator, 6, place the right hand on the numerator, 5, and say: 5 × 3 is 15. We clear the 5 and place the 15 in the numerator section. We now have on the abacus the numerator, 15, and the denominator, 18, or the fraction 15/18 which is the equivalent of 5/6.

EXAMPLES

(a) 3-1/3 + 4-5/6 = ____

(b) 7-3/5 + 4-3/4 = ____

(c) 8-2/5 + 3-4/9 = ____

(d) 4-5/7 + 3-5/8 = ____

(e) 11-5/6 + 7-4/9 = ____

(f) 15-2/3 + 4-5/6 = ____

ANSWERS

(a) 8-1/6

(b) 12-7/20

(c) 11-38/45

(d) 8-19/56

(e) 19-5/18

(f) 20-1/2

SUBTRACTION OF FRACTIONS

Examples in subtraction of fractions are set up as in addition of fractions. Again, we divide the abacus into three sections: — the whole numbers to the left of the second unit mark; the numerator to the left of the first unit mark; and the common denominator to the extreme right. In the beginning, addition and subtraction of fractions must first be done with simple examples having common denominators. In other words, examples with **direct** addition and subtraction make for an easier beginning.

Let us work the example: 9-7/8 — 5-3/8
Since the denominator 8 is common to both numbers, it is set to the extreme right. The whole number 9, of 9-7/8, is set to the left of the second unit mark, and the numerator, 7, of the fraction 7/8, is set to the left of the first unit mark. To proceed with the subtraction, the whole number, 5, subtracted from the whole number, 9, gives us 4, and the numerator, 3, of the 3/8, subtracted from 7, of 7/8, gives us 4. Taking into account the common denominator, 8, the answer reads 4-4/8 or 4-1/2.

To demonstrate how to work with fractions using a lowest common denominator, let us solve 7-5/8 — 5-2/3. First, we find the common denominator by multiplying the two denominators, 8 and 3. 8 × 3 is 24. We set the 24 to the extreme right. Resuming the setting of the example, we set the 7 of 7-5/8 in the whole number section which is to the left of the second unit mark from the right. Then we proceed to change the 5/8 to an equivalent fraction with the common denominator, 24. To do this, we set the 5 of 5/8 to the extreme left, skip two rods (to provide for space for two additional digits, if needed) and set the 8. As explained in a previous section, we multiply both the numerator and the denominator by the same number without changing the value of the fraction. In this case, the number is 3, obtained by dividing the denominator, 8, into the common denominator, 24. Therefore the fraction, 5/8, times 3/3, becomes 15/24. The denominator, 24, is already in its proper position; we need only to set the numerator, 15, in the numerator section which is to the left of the first unit mark.

Subtracting the whole number, 5, from the whole number, 7, gives us 2. We are now ready to subtract 2/3 from 15/24. Since we can not subtract 3rds from 24ths, we change the 2/3 to 24ths by multiplying both the numerator and denominator of 2/3 by 8 (obtained by dividing the 3 of 2/3 into 24). This gives us 16/24.

Since we can not subtract 16/24 from 15/24, we "borrow" a whole 1 from the whole number 2, which leaves us 1 in the whole number section. The "borrowed" 1, which is equivalent to 24/24 is added to 15/24, giving us 39/24. We are now able to subtract 16 of the 16/24 from 39 in the numerator section, resulting in 23/24. Our answer is 1-23/24.

EXAMPLES

(a) 9-5/7 − 5-3/7 = _____	(d) 9-4/9 − 7-3/4 = _____
(b) 11-4/15 − 9-7/15 = _____	(e) 14-1/2 − 7-7/12 = _____
(c) 120-7/8 − 50-3/12 = _____	(f) 75-1/2 − 67-3/4 = _____

ANSWERS

(a) 4-2/7
(b) 1-12/15 or 1-4/5
(c) 70-15/24 or 70-5/8
(d) 1-31/36
(e) 6-11/12
(f) 7-3/4

MULTIPLICATION OF FRACTIONS

In the multiplication of fractions, we divide the abacus into two sections: — The numerators on the left side of the abacus, and the denominators on the right.

Let us take the example 1/2 × 3/4. We set the numerator, 1, to the extreme left, and the denominator, 2, to the extreme right. To set the 3 of the 3/4, we skip two rods (to provide for space for additional digits, if needed) to the right of the first numerator, 1. To set the 4 of the 3/4, we skip two rods to the left of the first denominator, 2. On the left side of the abacus, we now have the numerators 1 and 3; and on the right side, we have the denominators, 2 and 4.

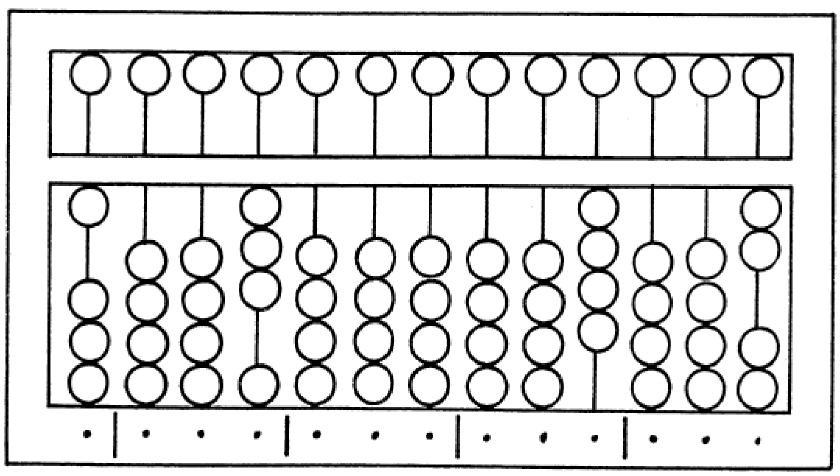


Fig. 18 (1/2 × 3/4)
Numerators on left; Denominators on right

In the multiplication of fractions, we multiply numerator by numerator, and denominator by denominator. In this case, to multiply the numerators, we place the right hand on the first numerator, 1, and then, with the right hand we check for the second numerator which is 3. As the right hand moves from the numerator, 1, to the numerator, 3, the left hand is placed on the 1. Then we say: 1 × 3 is 3, and we clear the 1 with the left hand. The product 3 remains as the new numerator.

In multiplying the denominators, we place the right hand on the first denominator, 4, and then with the right hand, we check for the second denominator, which is 2. As the right hand moves from the denominator, 4, to the denominator, 2, the left hand is placed on the 4. Then we say: — 4 × 2 is 8; we clear the 4 with the left hand, and with the right hand change the 2 to the product, 8, which is the new denominator. The answer is 3 in the numerator and 8 in the denominator, or the fraction 3/8.

Let us take another example: 3-1/2 × 1-1/7. Since these are mixed numbers, they must first be changed to improper fractions. 3-1/2 becomes 7/2. We set the numerator 7 to the extreme left, and the denominator, 2, to the extreme right. Then we change 1-1/7 to 8/7. We skip two rods after the first numerator, 7, and set the numerator, 8. Then we place the denominator, 7, two rods to the left of the denominator, 2. We check to see if we can cancel.

This example calls for an introduction to cancellation. We check to see if there is a number that can be divided evenly into one of the numerators and one of the denominators. With the right hand, we find the position of the first numerator, 7, then placing the left hand on that 7, we move the right hand to check in the denominator section for a number that can be evenly divided by 7. The right hand finds a 7 in the denominator section. 7 divided by 7 is 1, so the right hand changes the 7 to 1. We move the right hand to the numerator section where the left hand is resting on the 7, and we say: 7 divided by 7 is 1, and change that 7 to a 1 also.

The right hand checks the second numerator. It is 8. The left hand is placed on the 8, while the right hand moves to check the denominators in order to see if cancellation is possible. The first denominator is 1, and the second denominator is 2. Since 2 and 8 (a numerator and denominator) are evenly divisible by 2, we divide them by 2. 2 divided by 2 is 1, so the right hand changes the 2 to 1. Then the right hand returns to the 8 in the numerator, and we say: — 8 divided by 2 is 4, so we change the 8 to 4. We are now ready to multiply.

To multiply the numerators, we check the first numerator, 1, by placing the right hand on the 1. To check for the second numerator, we move the right hand to the 4, and place the left hand on the 1. We say: — 1 × 4 is 4. The left hand clears the numerator, 1, and the 4 remains as the new numerator in the numerator section.

To multiply the denominators, we check the first denominator, 1, by placing the right hand on the 1. To check for the second denominator, we move the right hand to the other 1, and place the left hand on the first denominator, 1. We say: — 1 × 1 is 1. The left hand clears the 1 on which it is resting, and the 1 remains as the new denominator in the denominator section. The answer is 4 in the numerator, and 1 in the denominator, 4/1, or the whole number 4.

EXAMPLES

(a) $1/3 \times 5/7 =$ _____

(b) $1\text{-}1/7 \times 1\text{-}1/8 =$ _____

(c) $2\text{-}1/6 \times 4\text{-}2/5 =$ _____

(d) $7/8 \times 4/9 =$ _____

(e) $2\text{-}1/3 \times 1\text{-}2/5 =$ _____

(f) $3\text{-}3/4 \times 2\text{-}1/5 =$ _____

ANSWERS

(a) 15/21

(b) 1-16/56 or 1-2/7

(c) 9-16/30 or 9-8/15

(d) 7/18

(e) 1-19/30

(f) 8-1/4

DIVISION OF FRACTIONS

Division of fractions is done in a similar manner as multiplication of fractions. However, the chief difference is that we invert the divisor before proceeding with the multiplication. Then we multiply numerator by numerator and denominator by denominator.

Since the division of fractions is the inverse of multiplication, we can refer to the division of fractions as the multiplication of the dividend by the **reciprocal** of the divisor. The reciprocal of a fraction is the inverse of that fraction. For example, the reciprocal of 2/3 is 3/2, and the reciprocal of 3 is 1/3. Having determined the reciprocal of the divisor, we write each division example as a multiplication example and then find the answer.

Let us divide 2/3 by 1/5. To set the dividend, 2/3, we set the numerator, 2, to the extreme left, and the denominator, 3, to the extreme right. To set the divisor, we first obtain the reciprocal of 1/5 by inverting the fraction giving us 5/1. We set the numerator, 5, two rods to the right of the numerator, 2, of the dividend. The 1 of the denominator in 5/1 is set on the third rod to the left of the 3.

We then proceed as in multiplication of fractions, i.e., multiplying the numerators (2×5) giving us 10 in the new numerator, and multiplying the denominators (1×3), giving us 3 in the new denominator. The answer is 10/3. Since 10/3 is an improper fraction, we divide the numerator, 10, by the denominator, 3, and the answer is 3-1/3. (In this case cancellation was not possible.)

EXAMPLES

(a) $1/2 \div 3/4 =$ _____

(b) $2\text{-}1/4 \div 1\text{-}2/7 =$ _____

(c) $5\text{-}2/3 \div 1\text{-}2/5 =$ _____

(d) $4/5 \div 2/5 =$ _____

(e) $1\text{-}1/8 \div 1/4 =$ _____

(f) $3\text{-}1/3 \div 2\text{-}1/2 =$ _____

ANSWERS

(a) 2/3

(b) 1-3/4

(c) 85/21 or 4-1/21

(d) 2

(e) 4-1/2

(f) 2-1/3

PER CENT

Since per cent indicates hundredths, per cent can be written as a fraction or a decimal. For example, to obtain 15% of 125, we first change 15% to a fraction, 15/100, or to a decimal, .15, and multiply. If we use the fraction, 15/100, as the multiplier, we reduce it first to 3/20 and set the fraction as in multiplication of fractions. (Set the numerator, 3, to the extreme left, and the denominator, 20, to the extreme right.) The 125 of the multiplicand, (in 125/1), is set three rods to the right of the numerator, 3; and the denominator, 1, is set three rods to the left of the denominator, 20. By cancellation and multiplication, we obtain the answer, which is 18-3/4.

If we use the decimal, .15, as the multiplier, we set it up on the abacus as in multiplication of decimals. We set the 15 to the extreme left and the 125 on the sixth, fifth, and fourth rods from the extreme right and multiply. The answer is 1875. But, since per cent is the same as hundredths, we point off two places in the answer, giving us 18.75. If, however, an example contains other decimal places, the two places for per cent are added to the number of those decimal places before pointing off.

When the per cent has a simple fractional equivalent such as 50% = 1/2, 25% = 1/4, 33-1/3% = 1/3, etc., it is easier and quicker to use the fractional equivalent as the multiplier.

EXAMPLES

(a) 17% of 215 = _____ (d) 25% of 210 = _____

(b) 12.3% of 47 = _____

(c) 6.05% of 103 = _____

(e) 62-1/2% of 184 = _____

(f) 87-1/2% of 168 = _____

ANSWERS

(a) 36.55

(b) 5.781

(c) 6.2315

(d) 52.5

(e) 115

(f) 147

SQUARE ROOT

The square root of a number is one of the two equal factors of the number. For example, the square root of 16 is 4 since 4 × 4 is 16, or 4 squared equals 16. Though there are many tables of squares and square roots, they are not always available, so it might be necessary to extract the square root on the abacus. The square root of a number is extracted by means of division. This is done in much the same way as in **long division**, with an exception that the divisor **changes** with each step. In this manual, however, instead of doubling the divisor, as in the conventional manner of extracting the square root, we **halve** the **dividend**, thereby working with smaller numbers.

Before proceeding with the solution of an example to find the square root of a number, we separate the digits in the number into groups of two from right to left starting with the decimal point. There will then be as many digits in the square root as there are groups of two digits in the original number. When the remainder is one-half of the square of the final digit of the square root, the original number is a perfect square.

To illustrate the procedure, let us extract the square root of 1225. To make provision for one decimal, we count six rods from the right to start the setting of 1225. We separate the digits in the number into groups of two, from **right to left** starting from the decimal point, but we work the example from **left to right**. In this example there are two groups, 12 and 25. Therefore, there will be two digits in the square root. For the first digit in the square root of 1225, we extract the square root of the largest perfect square within the first group, (12).

To extract the square root of the 12, we say: —"What is the greatest square in 12?" The greatest square within 12 is 9, and the square root of 9 is 3. We proceed as in division by dividing, multiplying, and subtracting. We set the 3 immediately to the left of the 1 in the 12 group. 3 × 3 equals 9. After subtracting 9 from 12 which gives us 3, we now have 325 on the abacus. We divide the 325 in half, which gives us 162.5. Then we divide 3 into 16 and get 5 and something remaining. We set 5 to the left of the 16. 5 × 3 equals 15. Subtracting 15 from 16 leaves us 1. We now have 12.5 in the new dividend and 35 in the quotient. Squaring the 5 of 35, gives us 25. Dividing the 25 by 2 gives us 12.5. Since the 12.5 on the abacus is exactly 1/2 of the square of 5 (25), we know that 1225 is a perfect square, and the square root of 1225 is 35.

Let us extract the square root of 86 which is not a perfect square, and will therefore require the use of decimals. After the 86 is set on the sixth and fifth rods, we have four zeros which make it possible to carry out the answer to two decimal places. The greatest square root in 86 is 9 (9 × 9 = 81). We set the 9 to the left of the 8 in 86. Subtracting 81 from 86, the answer is 5. We now have 5.0000. Dividing 5.0000 in half, we get 2.5000. Dividing 2.5 by 9, we set .2 which we set in the quotient to the right of the 9. Multiplying .2 by 9 gives us 1.8 which we subtract from 2.5, giving us .7. We now have .7000 on the abacus. We divide 92 into .7000 and get approximately 7. Placing the 7 to the right of 9.2, gives us the answer 9.27.

Let us take the example 7396. We separate the number from right to left into groups of two. The two groups are 73 and 96. The greatest square root in the first group, 73, is 8 ($8 \times 8 = 64$). We subtract 64 from 73, giving us 9. We now have 996. We divide that number in half and get 498. When we divide 8 into 49, we get 6 and something remaining, ($8 \times 6 = 48$). We subtract 48 from 49, leaving us 1. We now have 18. If we square the 6 which had been set to the right of the 8 in the quotient, and divide the square in half, we get 18. ($6 \times 6 = 36$; 36 divided by $2 = 18$). Since 18 is 1/2 of the square of 6, the final digit of the square root, 86, we see that 7396 is a perfect square, and the square of it is 86.

Let us extract the square root of 841. We divide 841 into two groups from right to left, giving us 41 and 8. For the first digit in the square root of 841, we take the square root of the largest perfect square within 8, the first group. The greatest perfect square within 8 is 4. The square root of 4 is 2. Since the 2 is smaller than 8, we skip a rod from the 8 and set the 2. We square 2, giving us 4 which we subtract from the 8, leaving 4. We now have 441 which we divide in half, giving us 220.5. Dividing the 2 into the 22 of 220.5, we get 11. However, since we must have only one digit in the square root, for every group in the number, we take the highest possible one-digit number, 9, and place it next to the 2, giving us 29. We multiply 9 by 2, giving us 18 which we subtract from 22, leaving us 4. We now have 40.5 on the abacus. Squaring the 9 and dividing it in half, we obtain 40.5 ($9 \times 9 = 81$; 81 divided by $2 = 40.5$). Again we see that this is a perfect square.

Note: — When the remainder on the abacus is one half of the square of the final digit of the square root, the original number is a perfect square.

Here is a five-digit number: — 17689. In a five-digit number, it is often easier to start by finding the square root of the first two groups of the number. We divide 17689 into 3 groups moving from right to left (1, 76, 89). We extract the square root of the first two groups, 1, 76. The square root of 176 is 13 (by trial and error). We set 13 to the left of 176. Squaring 13 gives us 169. We subtract 169 from 176 which is 7, giving us 789 on the abacus. We halve 789, giving us 394.5. Dividing 13 into 39 of 394.5 gives us 3 which we set to the right of the 13. Subtracting 39 from the 39 of 394.5 leaves 4.5 on the abacus. If we square the final 3 of the 133 in the quotient giving us 9, and halve the 9, we get 4.5. Since 4.5 is half of the square of the final 3 in the quotient, we know that 17689 is a perfect square, and that 133 is the square root of 17689.

Another method for extracting square root is that adapted from the Japanese by T. V. Cranmer. This is the **successive subtraction method**. He gives no explanation for the operation involved but suggests that the mechanics of the method be taught per se. The procedures of this method may best be illustrated by an example given by Mr. Cranmer:

Take the number 1225 as the beginning point from which we want to find the square root. We set the number 1225 on the right end of the abacus, without separation, although we consider it mentally as two groups of two digits each, 12 and 25. Then, from the first group, 12, we subtract, in succession, the odd numbers 1, 3, 5, etc. from the successive differences until no further subtractions are possible.

Each time we make a subtraction, we tally by pushing up a bead at the extreme left end of the abacus. The number of beads pushed up is the number used as a digit in the root. Thus, we find in this example that after the third subtraction, the process can not be continued. ($12 - 1 = 11$; $11 - 3 = 8$; $8 - 5 = 3$) and so 3 is the first digit of the root we are looking for. It will be found that in the use of the odd number series 1, 3, 5, 7, 9, 11, 13, 15, 17 it will never be necessary to go beyond 17 since this would be the ninth subtraction, and 9 is the last one-digit number to be used as the digit in the root.

Now to return to the example of extracting the square root of 1225: after the third subtraction in the first group, we have a 3 remaining. This 3 must now be considered as the first digit in the second group so that we now have a three-digit number to work with, viz.: 325. To determine the numbers to be subtracted in this group, we first **double** the digit in our first answer mentally ($2 \times 3 = 6$) and annex the odd numbers 1, 3, 5, etc. to 6, so that we first subtract 61, then 63, then 65, etc. until no further subtraction is possible, remembering to tally after each subtraction by pushing up a bead in the column next to the previous tally column. By this process we find that 5 subtractions are possible, and that 5 is therefore the second digit in our square root. Thus the square root of 1225 is 35.

Let us take another example: Find the square root of the number 163216. After setting 163216 on the right end of the abacus, we divide the number into two-digit groups beginning at the decimal end and working toward the left. This gives us three groups: 16, 32, 16. In the first group we subtract 1, 3, 5, 7 ($16 - 1 = 15$, $15 - 3 = 12$, $12 - 5 = 7$, $7 - 7 = 0$) remembering to tally after each subtraction by pushing up a bead at the left end of the abacus. Thus we arrive at 4 as the first digit in the answer.

To determine the numbers to be subtracted in the second group, we double 4 and annex the odd numbers in succession. Thus $2 \times 4 = 8$, so we should first subtract 81 from the second set of two-digit numbers; but 81 is too large to subtract from 32, so the second digit in the answer is 0, making the partial answer 40. Subtractions must therefore proceed from the next two sets of digits, i.e., from 3216. We double 40 and annex 1, giving us 801 which we subtract from 3216. The successive subtrahends are 803, 805, 807, making four subtractions which we tally by pushing up beads in the third column from the left. This gives us 4 as the third digit in the final answer which is 404. Note: Again we see that the number of subtractions tally with the digit used in the answer.

EXERCISES

(a) 81 ____ (d) 1936 ____

(b) 784 ____ (e) 2601 ____

(c) 1225 ____ (f) 1550 ____

ANSWERS

(a) 9	(d) 44
(b) 28	(e) 51
(c) 35	(f) 39.37

CONCLUSION

This manual is offered to both teachers and students in order to advance the use of the Cranmer Abacus.

The writer has found that mastery of this tool has been of inestimable help in increasing the speed and accuracy of mathematical computation among her students at the Overbrook School for the Blind in Philadelphia, and other schools in other parts of the country. Public school teachers attending seminars conducted by the author in colleges in Philadelphia, Pa., the South and Midwest attest to the fact that sighted, as well as blind, pupils find that the use of the abacus makes arithmetical computation easier and more accurate.

This manual was written in the hope that these step-by-step explanations will make it easier for teachers to learn the procedure, and to facilitate the teaching of the operations to the students. With the increasing use of the abacus, it is also hoped that publishers will recognize the need of putting into print a manual such as this.

The enthusiastic teacher will find a ready response from the students to whom a new world of mathematics is opened through the use of the Cranmer Abacus.